# Math 242 Final, Spring 2024

**Name:** ______________________________ **Section:** ______

**Instructor:** __________________________________________

**TA:** __________________________________________________

| Question | Points | Score |
|---|---|---|
| 1 | 11 | |
| 2 | 12 | |
| 3 | 12 | |
| 4 | 16 | |
| 5 | 8 | |
| 6 | 12 | |
| 7 | 12 | |
| 8 | 10 | |
| 9 | 12 | |
| 10 | 10 | |
| 11 | 10 | |
| 12 | 10 | |
| 13 | 9 | |
| Total: | 144 | |

- You may not use notes or calculators on the test. No internet or cell phone access. No electronic devices of any kind.
- Please ask if anything seems confusing or ambiguous.
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- You must show all your work. Please clearly indicate your final answer (e.g. by circling it).
- Organize your work neatly and legibly in the spaces provided under each problem.
- You do not need to simplify your answers unless explicitly told to do so.
- Clearly cross-out scratch work.
- You have exactly 2:00 hours to complete this Exam. Good luck!

1. Let $f(x) = \sqrt{9 - x^3}, \quad x \geq 0.$.

(a) (3 points) What is the domain of $f^{-1}(x)$?

(b) (3 points) Find the value of $x$ such that $f(x) = 1$.

(c) (5 points) Compute $(f^{-1})'(1)$.

2. Compute the derivatives of the following functions. **No need to simplify your answers.**

(a) (6 points) $f(x) = (\sin x)^x$. (Hint: Use properties of logarithmic functions.)

(b) (6 points) $f(x) = \sin^{-1}(5^x)$.

3. Compute the following limits.

(a) (8 points) On this part must justify your solution using algebraic manipulations and/or l'Hôpital's rule for full credit.

$$\lim_{x\to 0} \frac{e^x - \sin(x) - \cos(3x)}{x^2}.$$

(b) (4 points) Consider the following limit.

$$\lim_{n\to\infty} \left(1 + \frac{5}{n}\right)^n.$$

Which of the following equals this limit? Circle your answer. (On this part you do NOT need to justify your work).

$\mathbf{1}$ $\mathbf{e^{-5}}$ $\infty$ $\mathbf{e^{5}}$ $\mathbf{0}$

4. Compute each integral.

(a) (6 points) $\int \sin^2(x)\cos^3(x)dx$.

(b) (10 points) $\displaystyle\int_3^{\infty} \frac{3}{x(3x+1)}dx.$

5. (8 points) **Calculate the sum** of the series, if it is convergent. If it diverges, explain why.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3}{7^n}.$$

6. Determine whether each series below is absolutely convergent, conditionally convergent or divergent. State which test(s) you use and justify the answer.

(a) (6 points) $\sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2}+\frac{1}{n}\right)^n$.

(b) (6 points) $\displaystyle\sum_{n=2}^{\infty} \frac{1+e^n}{e^n - n^2}$.

7. Determine whether each series below is convergent or divergent. State which test(s) you use and justify the answer.

(a) (6 points) $\displaystyle\sum_{n=3}^{\infty} \frac{3n^3+3}{n^7-2n^2-11}$.

(b) (6 points) $\sum_{n=1}^{\infty} \frac{n^{10}}{(2n+1)!}$.

8. (10 points) Find the interval of convergence for the following power series.

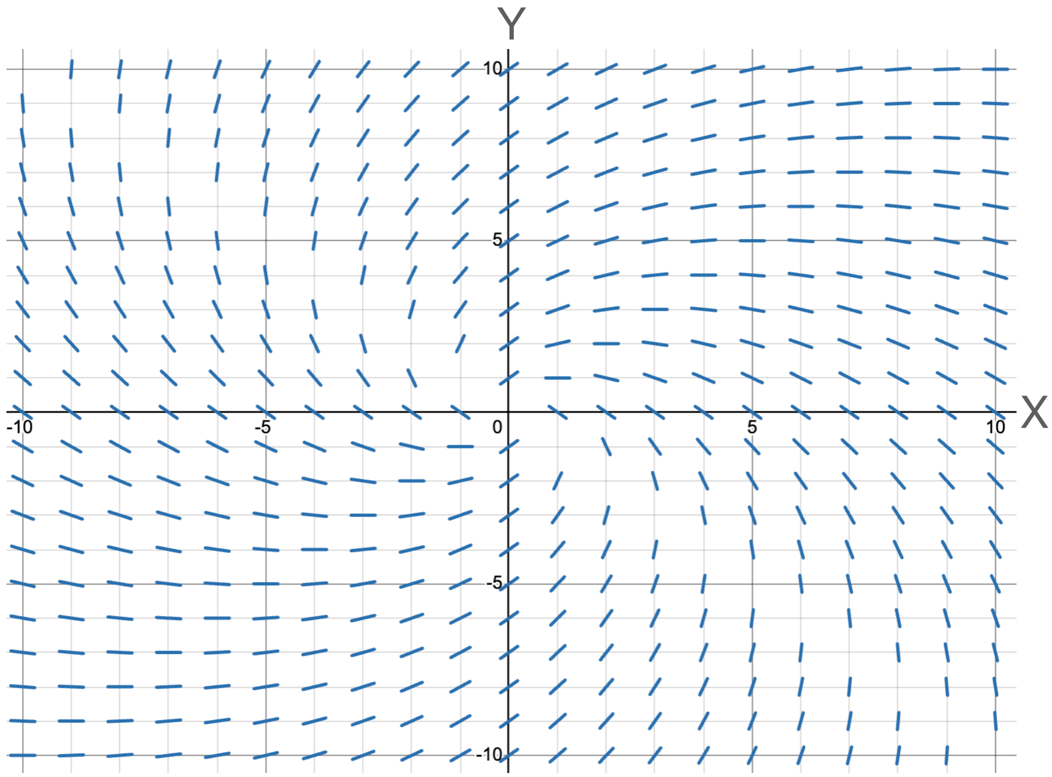
$$\sum_{n=1}^{\infty} \frac{(2x+1)^n}{3n+5}.$$

Make sure to check the endpoints.

9. (a) (6 points) Find a power series representation for the function $\dfrac{1}{1-3x^2}$. (Hint: you may use a well-known power series to help solve this problem.)

(b) (6 points) Use part (a) to express the integral $\int \frac{1}{1-3x^2}\, dx$ as a power series.

10. Consider the direction field pictured below. (The horizontal and vertical gridlines may be hard to see in the photocopy, but they are not necessary to solve the problem.)




(a) (5 points) Which of the differential equations below matches this direction field? No justification needed, just circle your answer.

(a) $\dfrac{dy}{dx} = \dfrac{x}{y}$, (b) $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, (c) $\dfrac{dy}{dx} = \dfrac{x+y}{y}$, (d) $\dfrac{dy}{dx} = \dfrac{y}{y+x}$.

(b) (5 points) Sketch the solution to this differential equation that satisfies the initial condition $y(0) = 5$ on the direction field.

11. (10 points) Find a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for $y$ for full credit.

$$\frac{dy}{dx} = e^{-y}(2x+3) \text{ and } y(1) = \ln(6).$$

12. (10 points) Find a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for $y$ for full credit.

$$\frac{dy}{dx} + 3x^2 y = x^2 \text{ and } y(0) = \frac{7}{3}.$$

13. Determine whether each of the following statements is true or false. **No need to justify your answer.**

(a) (3 points) $\int \frac{1}{1+5x^2} dx = \frac{1}{\sqrt{5}} \tan^{-1}(x) + C$.

(b) (3 points) The series $\sum_{n=1}^{\infty} (-1)^n \sin(n+1)$ conditionally converges.

(c) (3 points) The sequence $(a_n)$ converges if $a_n = \frac{1+n}{2-5n}$

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**Formula sheet**

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{(true for } -1 < x < 1\text{)}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \quad \text{(true for all } x\text{)}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{(true for } x < -1 \text{ and } x > 1\text{)}$$

- Pythagorean identities (true for all $x$ where the functions involved are defined).

$$\sin^2(x) + \cos^2(x) = 1, \quad \tan^2(x) + 1 = \sec^2(x), \quad 1 + \cot^2(x) = \csc^2(x).$$

- Reduction of power formulas / double angle formulas for sine and cosine (true for all $x$).

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)), \quad \sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

- Addition formulas for sine and cosine (true for all $x$ and $y$).

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of tangent and secant.

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

- Trapezoidal Rule and Simpson's Rule:

$$T_n = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S_n = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error Bound for Trapezoidal Rule and Simpson's Rule:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}, \qquad \text{where } |f''(x)| \leq K \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \leq \frac{M(b-a)^5}{180n^4}, \qquad \text{where } |f^{(4)}(x)| \leq M \text{ for all } x \text{ in } [a,b]$$

- Standard power series expansions (centered at $a = 0$).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(valid for all } x\text{).}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(valid for all } x\text{).}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{(valid for all } x\text{).}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad \text{(valid for } |x| < 1\text{).}$$

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n \qquad \text{(valid for } |x| < 1\text{).}$$
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

- Error estimate for approximations by Taylor polynomials.
  Say $f(x)$ is a function with derivatives of all orders on an interval $[b,c]$, and $a$ is a point in $[b,c]$. Say $T_N(x)$ is the $N^{\text{th}}$ Taylor polynomial for $f(x)$ centered at $a$, and $R_N(x) = f(x) - T_N(x)$ is the error when approximating $f(x)$ by $T_N(x)$. Then for all $x$ in $[b,c]$

$$|R_N(x)| \leq \frac{M_{N+1}|x-a|^{N+1}}{(N+1)!},$$

  where $M_{N+1}$ is the largest value taken by the $(N+1)^{\text{st}}$ derivative of $f(x)$ on $[b,c]$.