# ANALYSIS QUALIFYING EXAM - AUGUST 2024

Attempt the following six problems. Please note the following:

- Throughout this exam, $L^p(X)$ denotes the $L^p$-space of a measure space $(X, \mathcal{M}, \mu)$ with the associated norm of a function being $\|f\|_p$. Subsets of $\mathbb{R}^d$ are equipped with the Lebesgue sigma algebra and Lebesgue measure unless otherwise stated.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

(1) Let $\mathcal{P}$ be the power set of $[0,1]$, and let $\mu$ denote the counting measure on the measurable space $([0,1],\mathcal{P})$. Define

$$\mathcal{M} = \{E \subseteq [0,1] : E \text{ or } E^c \text{ is countable}\}$$

(here $E^c = [0,1] \setminus E$).

(a) Prove that $\mathcal{M}$ is a $\sigma$-algebra.

Let $\mu_0$ be the counting measure on the measurable space $([0,1],\mathcal{M})$.

(b) Give an example of a function $f : [0,1] \to \mathbb{R}$ that is $\mathcal{P}$-measurable, but not $\mathcal{M}$-measurable.

(c) Show that a function $f : [0,1] \to \mathbb{R}$ that is integrable with respect to $(\mathcal{P},\mu)$ is also integrable with respect to $(\mathcal{M},\mu_0)$.

(2) Let $f : [0,1] \to \mathbb{R}$ be Lebesgue integrable, and suppose that $\lim_{x\to 1} f(x) = a$ for some $a \in \mathbb{R}$. Prove

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\,dx = a.$$

*Hint: do the case $a = 0$ first.*

(3) Prove or disprove the existence of a Lebesgue measurable set $E \subseteq [0,1]$ with the property that $m(E \cap [0,x]) = x/2$ for every $x \in [0,1]$.

(4) Let $f_n \in L^1([0,1])$ and $C > 0$ be such that $\|f_n\|_1 \leq C$ for all $n$, and suppose that $f_n \to f$ pointwise almost everywhere.

(a) Prove that

$$\int f_n g\; dm \to \int fg\; dm$$

for every $g \in L^\infty([0,1])$.

(b) Is the analogous statement with $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ replacing $L^1([0,1])$ and $L^\infty([0,1])$ true? Give a proof or a counterexample.

(5) Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Define $f * g : \mathbb{R} \to \mathbb{C}$ by

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$$

*You may assume without proof that $(f * g)(x)$ is well-defined for almost every $x \in \mathbb{R}$, and measurable as a function of $x$.*

Show that $f * g$ is in $L^p(\mathbb{R})$, and that $\|f * g\|_p \leq \|f\|_1 \|g\|_p$.

(6) Recall that a function $f : [0,1] \to \mathbb{R}$ is *absolutely continuous* if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $n$ and any

$$0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leq 1$$

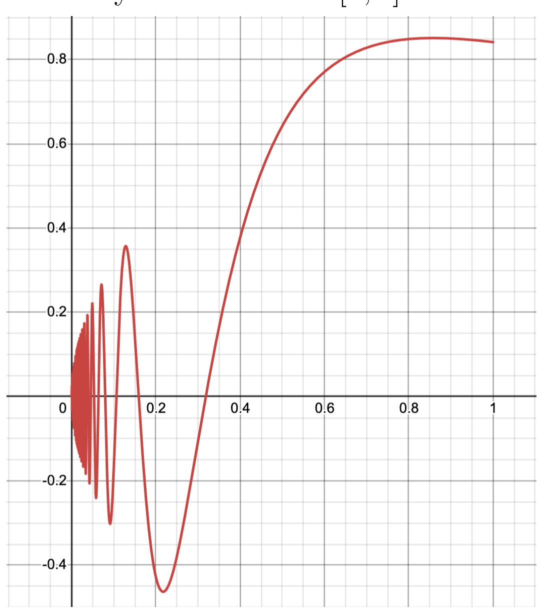
with

$$\sum_{i=1}^{n} |b_i - a_i| < \delta$$

we have

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon.$$

Let $\alpha \in (0,1)$. Show that the function defined by $f(x) = x^\alpha \sin(1/x)$ for $x > 0$ and $f(0) = 0$ is not absolutely continuous on $[0,1]$. Here is a picture for $\alpha = 1/2$:




*Hint: do not try to use sophisticated theorems! - do the estimates directly.*