ANALYSIS QUALIFYING EXAM, SPRING 2024

- Throughout this exam (X, μ) denotes an arbitrary measure space, and $L^p(\mu)$ the corresponding Lebesgue space. Also, m denotes the Lebesgue measure in $\mathbb{R}^d$, and $L^p(E)$ denotes the Lebesgue space (real-valued functions) of a subset E of $\mathbb{R}^d$ with respect to m.
- Partial credit will be given for partially correct solutions, even if incomplete.
- The parts of problems are not equally difficult, and will not be weighted equally.
- Good luck!

Problem 1. Let E be the set of real numbers in $(0, 1)$ that do not have a 2 in their decimal expansion.

a. Show that E is measurable.

b. What is the Lebesgue measure of E?

Problem 2. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous. Suppose that $\frac{\partial f}{\partial x}$ exists and is continuous on $[0, 1] \times [0, 1]$. Show that the function $g : (0, 1) \to \mathbb{R}$ defined by

$$g(x) = \int_0^1 f(x, y)\, dy \qquad (x \in (0, 1))$$

is differentiable and that

$$g'(x) = \int_0^1 \frac{\partial f}{\partial x}(x, y)\, dy.$$

Problem 3. Let $f \in L^2((0, \infty))$. Define $Hf : (0, \infty) \to \mathbb{R}$ by

$$Hf(x) := \int_0^\infty \frac{f(y)}{x + y}\, dy \qquad (x \in (0, \infty)).$$

a. Show that Hf is measurable.

b. Prove that there is a constant C such that

$$\|Hf\|_2 \le C\|f\|_2.$$

Here $\| \cdot \|_2$ denotes the L^2-norm on $(0, \infty)$.

Problem 4. Let $1 < p < \infty$, and let (f_n) be a sequence of functions in $L^p([0,1])$ that converges almost everywhere to a function $f \in L^p([0,1])$. Suppose in addition that there is a constant M such that $\|f_n\|_p \le M$ for all n.
Show that for each $g \in L^q([0,1])$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg.$$

Problem 5.

a. Show that if f is a continuously differentiable function on $[0,1]$, then there is a sequence of polynomials (P_n) such that $P_n \to f$ and $P_n' \to f'$ uniformly on $[0,1]$.

b. Does this remain true if $[0,1]$ is replaced by $\mathbb{R}$? Explain.

Problem 6. Suppose that μ is finite, and let $T : X \to X$ be measurable such that $\mu(T^{-1}(E)) = \mu(E)$ for every measurable set $E \subset X$. Let $A \subset X$ be measurable.

a. Show that if $B := A \setminus \bigcup_{n\in\mathbb{N}} T^{-n}(A)$, then the sets $T^{-n}(B), n \in \mathbb{N}$, are pairwise disjoint. Here $T^{-n}(E)$ denotes the preimage of a set $E \subset X$ under the n-fold composition of T with itself.

b. Deduce that almost every $x \in A$ is *recurrent*, i.e., $T^n(x) \in A$ for some $n \in \mathbb{N}$.