MOCK ANALYSIS QUALIFYING EXAM 2

Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denoted the Lebesgue measure of a set A by $m(A)$.

(1) Let $A \subset [0, \infty)$ and let $B = \{a^2 \mid a \in A\}$

(a) Show that if $m(A) = 0$, then $m(B) = 0$.

(b) Suppose $m(A) < \infty$, it it true that $m(B) < \infty$? If so, prove it. Otherwise, provide a counterexample.

(2) Let (f_n) be a sequence of functions converging in L_p to f.

(a) Show that a subsequence (f_{n_k}) converges to f almost everywhere.

(b) Give an example of a sequence of functions which converge to 0 in L_2 but not almost everywhere.

(3) (a) Suppose $f \in L_4([0, 1])$. Show that $f \in L_2([0, 1])$ and $\|f\|_2 \leq \|f\|_4$.

(b) Show that there does not exist a constant C so that for all $f \in L_4([0, 1])$ $\|f\|_4 \leq C\|f\|_2$.

(c) Suppose $f \in L_4([0, 1])$ and $\|f\|_4 \leq C\|f\|_2$. Find a constant C_1 depending only on C so that $\|f\|_2 \leq C_1\|f\|_1$. (*Hint:* $\frac{1}{3} + \frac{2}{3} = 1$)

(4) Give an example of a continuous and integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ with the following property: there are infinitely many $y \in \mathbb{R}$ such that the slice $g^y : \mathbb{R} \to \mathbb{R}$ defined by $g^y(x) = f(x, y)$ is not integrable. Justify your example.

(5) Suppose (u_k) is a sequence of differentiable functions in $L_2(\mathbb{R})$ satisfying

- there is $u \in L_2(\mathbb{R})$ so that $\|u_k - u\|_2 \to 0$
- there is $v \in L_2(\mathbb{R})$ so that $\|u_k' - v\|_2 \to 0$

If u is differentiable, show that $u' = v$ almost everywhere.

(6) For what values of $p \in (0, \infty)$ is

$$f(x, y) = \frac{x - y}{(x + y)^p}$$

Lebesgue integrable over $[0, 1]^2$?