MOCK ANALYSIS QUALIFYING EXAM 5

Attempt the following six problems. Please note the following:

- Throughout the exam, unless indicated otherwise, integration is with respect to Lebesgue measure.
- We denote the Lebesgue measure of a set A by $m(A)$.

(1) Let m denote Lebesgue measure on $\mathbb{R}^3$, and let

$$E := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 + 4z^2 = 1\}.$$

Define a subset A of E to be *E-measurable* if

$$A_0 := \{ta \in \mathbb{R}^3 \mid t \in [0, 1], a \in A\}$$

is measurable in $\mathbb{R}^3$, and for an E-measurable set $A \subseteq E$ define

$$m_E(A) := m(A_0).$$

(a) Show that the collection of E-measurable sets is a σ-algebra, that m_E is a measure on E, and that every continuous function $f : E \to \mathbb{R}$ is measurable for this σ-algebra.

(b) Compute $m_E(E)$ (you may assume the measure of the unit ball in $\mathbb{R}^3$ is $\frac{4}{3}\pi$).

(2) (a) Say $f : [0, 1] \to \mathbb{R}$ is continuous, and differentiable on $(0, 1)$ with bounded derivative. Show that f has bounded variation.

(b) For each $a > 0$, define

$$f_a : [0, 1] \to \mathbb{R}, \quad f_a(x) = \begin{cases} x^a \sin(x^{-2}) & x > 0 \\ 0 & x = 0 \end{cases}.$$

For which values of a does f_a have bounded variation? Justify your answer.

(3) Let $f : \mathbb{R} \to (0, \infty)$ be defined by $f(x) = x^{-1/2}\chi_{(0,1)}(x)$. Let $(q_n)_{n=1}^{\infty}$ be an enumeration of the rationals, and define $F : \mathbb{R} \to [0, \infty]$ by

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - q_n).$$

Show that the series defining F converges to a finite number almost everywhere, but that F is unbounded on any (non-empty) interval.

(4) Let $p > q$ be fixed numbers in $[1, \infty]$. Give a proof or a counterexample to each of the statements below.
(a) If $f \in L^p(\mathbb{R})$ has finite measure support, then $f \in L^q(\mathbb{R})$.

(b) If $f \in L^q(\mathbb{R})$ has finite measure support, then $f \in L^p(\mathbb{R})$.

(c) If $f \in L^p(\mathbb{R})$ is bounded, then $f \in L^q(\mathbb{R})$.

(d) If $f \in L^q(\mathbb{R})$ is bounded, then $f \in L^p(\mathbb{R})$.

(5) Let X be a measure space. Prove that the (integrable) simple functions are dense in $L^\infty(X)$ if and only if X has finite measure.

(6) For each n, let $p_n \in [1, \infty]$. Set

$$E := \left\{ (v_n)_{n=1}^{\infty} \mid v_n \in L^{p_n}(\mathbb{R}) \text{ and } \sum_{n=1}^{\infty} \|v_n\|_{p_n} < \infty \right\}$$

with the norm

$$\|(v_n)\| := \sum_{n=1}^{\infty} \|v_n\|_{p_n}$$

(you may assume without proof that E is a vector space and this is a norm).
(a) Show that E is complete.

(b) Let $v : \mathbb{R} \to \mathbb{R}$ be measurable and non-zero on a set of positive measure. Is it possible that the constant sequence $(v, v, v, v, ...)$ is in E (for some sequence (p_n))? Give an example of such an E and v, or a proof that no such v can exist.