APPLIED MATH QUALIFYING EXAM
JANUARY 2019

Solve all six problems. You have 4 hours. Good luck!
You need to demonstrate prociency in each area

Problem 1. Recall the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y-x),\\ \dot{y} &= = rx - y - xz,\\ \dot{z} &= xy - bz,\end{aligned}$$

where r, σ, b are positive constants. For $0 < r < 1$, show that the origin is globally asymptotically stable by considering a function $V_1 = \alpha x^2 + \beta y^2 + \gamma z^2$, for a suitable choice of constants α, β, γ. For $r \geq 1$, show by considering the function $V_2 = rx^2 + \sigma y^2 + \sigma(z-2r)^2$ that all trajectories eventually enter and then remain within a bounded region of phase space.

Problem 2. Consider the dynamical system in polar coordinates

$$\begin{aligned}\dot{r} &= r(\mu + 2r^2 - r^4),\\ \dot{\theta} &= 1 - \nu r^2 \cos\theta.\end{aligned}$$

Find the conditions on the parameters μ and ν under which there are zero, one, and two periodic orbits. For $\nu = 0$, deduce the stability of these orbits and show the results in the (μ, r) plane. For $\nu = 1/2$, describe the types of bifurcations that occur as μ is varied.

Problem 3. The map $F(x, \mu) = 1 - \mu x^2$ has a superstable 3-cycle at a certain parameter value μ_c. Find a cubic equation for this μ_c.

Problem 4. Consider the 1-d map

$$x_{n+1} = f(x_n) = x_n + a\cos(x_n)\sin(x_n)$$

1. Find the fixed points x^* of the map above in the interval $[0;\pi[$.
2. Determine the derivative of the map.
3. For the fixed point x_2^* (where $x_1^* < x_2^*$) determine the value of a, denoted by a_2, where the fixed point undergoes period doubling.
4. For the other fixed point x_1^* a period doubling occurs at $a_1 = -2$. Consider a small variation around x_1^*, i.e. $x_1^* = \varepsilon$ where $\varepsilon << 1$, and show that a two cycle for the map exists (for $a = a_1$ expand the map above around in x_n).

Problem 5. Consider the following heat problem on a rod of length L with source distribution $F(x,t)$ and initial condition $g(x)$

$$u_t = u_{xx} + F(x,t) \qquad x \in (0,L),\ t > 0.$$

$$u(x,0) = g(x), \qquad x \in (0,L).$$

1. Suppose that the rod is insulated at $x = 0$ and has a fixed temperature of 0 at $x = L$. Give the boundary conditions for this problem.
2. Suppose the rod contains a radioactive substance that is emitting heat at a constant rate $F(x,t) = C$. Give a formal solution with this source and the boundary conditions of (a).
3. Use the energy integral $E(t) = \int_0^L u^2 dx$ to show that this solution is unique.

Problem 6. Suppose A and B are $n \times n$ matrices with real entries and that B has rank 1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ where $f(t) = \det(A + tB)$ is a line (i.e, it has form $f(t) = mt + b$).