SAMPLE APPLIED MATH QUALIFYING EXAM 2

1. For the motion of a damped pendulum, show that the governing equation is of the form $\ddot{x} + b\dot{x} + c\sin x = 0$, $b > 0, c > 0$, stating any approximations you make. Perform linear stability analysis and classification of the fixed points and sketch the phase portrait for different cases that can occur depending on the parameters b and c.

2. Consider two systems on the plane $\mathbb{R}^2$:

   $$\dot{x} = f(x), \quad \dot{x} = g(x)$$

   where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are C^1 and perpendicular (i.e. $\langle f(x), g(x) \rangle = 0$ for all $x \in \mathbb{R}^2$).

   Prove that if one of the systems has a (nontrivial) periodic orbit, then the other has fixed point.

3. Consider the planar system given in polar coordinates by

   $$\begin{aligned} \dot{r} &= r(\mu + 2r^2 - r^4), \\ \dot{\theta} &= 1 - \nu r^2 \cos\theta. \end{aligned}$$

   (a) Find the conditions on parameters μ and ν under which there are zero, one, and two periodic orbits.

   (b) Fix $\nu = 0$ and perform linear stability analysis of these orbits.

   (c) Describe the types of bifurcation that occur as one of the parameters μ and ν is varied while the other is kept fixed.

4. Consider the following planar system

   $$\begin{aligned} \dot{x} &= \frac{1}{2}x + y + x^2 y, \\ \dot{y} &= x + 2y + \lambda y + y^2, \end{aligned}$$

   where $\lambda \in \mathbb{R}$ is a parameter. Use the center manifold to describe bifurcations near the origin.

5. Consider a discrete dynamical system $x \mapsto g_\mu(x)$ where $g_\mu : \mathbb{R} \to \mathbb{R}$ is the tent map with parameter μ, that is

   $$g_\mu(x) = \begin{cases} \mu x, & x < \dfrac{1}{2} \\ \mu(1-x), & x \geq \dfrac{1}{2}, \end{cases}$$

   and assume $\mu > 1$.

(a) Find fixed points of g_μ and determine their stability.

(b) Compute the Lyapunov exponent of g_μ.

(c) Compute the escape set, E, of the given dynamical system. That is, compute the set of all points $x \in \mathbb{R}$ such that the sequence $g_\mu^n(x)$, $n = 1, 2, \ldots$ is unbounded.

(d) Construct a one-to-one function from $\Lambda = \mathbb{R} \setminus E$, the complement of E, to $\Sigma = \{0, 1\}^{\mathbb{N}}$, the set of all sequences of 0's and 1's.

6. Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is continuous and bounded. Show that

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) g(y)dy$$

satisfies the following conditions:

(a) $u(x,t) \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(b) $u_t - \Delta u = 0$ for $x \in \mathbb{R}^n$ and $t > 0$.

(c) $\lim_{y \to x, t \to 0+} u(y,t) = g(x)$.

Use the above results to show that for initial heat distribution $g(x) \geq 0$ and not identically 0, the temperature $u(x,t)$ is greater than or equal to 0 for all $t > 0$. Explain why this shows **infinite propagation speed** for disturbances.

7. Consider the $n \times n$ matrix $\mathbf{A}$ obtained by discretizing $\frac{d}{dx}$ over $[0,1]$ with central differences:

$$\mathbf{A} = \frac{1}{2(n+1)} \begin{pmatrix} 0 & 1 & 0 & & 0 \\ -1 & 0 & 1 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & -1 & 0 & 1 \\ 0 & & 0 & -1 & 0 \end{pmatrix}$$

(a) For which values of n is $\mathbf{A}$ singular and for which values is it non-singular?

(b) Prove that all the eigenvalues of $\mathbf{A}$ are of the form $\lambda = it$, where $t \in [-\frac{1}{n+1}, \frac{1}{n+1}]$.