Topology qualifying exam sample

The exam consists of eight questions split into parts. You have four hours to complete it. In order to pass, you must exhibit substantial knowledge and ability in each of the following fields: general topology, homotopy theory and covering spaces, and homology theory.

1.

(a) Give an example of two topological spaces X, Y and a continuous bijection $f : X \to Y$ that is not a homeomorphism.

(b) Show that if X is compact and Y is Hausdorff, then every continuous bijection between them is a homeomorphism.

2.

(a) Give the definitions of the following terms: separable space, second countable space.

(b) Show that a separable metric space is second countable.

3. Let $X = \mathbb{R}^2 - \mathbb{Q}^2$. Show that X is path connected, and that $\pi_1(X)$ is uncountable.

4. Let $\mathbb{RP}^n$ be the real projective n-space, $n \geq 2$, and $T^m = S^1 \times \ldots \times S^1$ (m factors) be an m-dimensional torus. Show that any continuous map $f : \mathbb{RP}^n \to T^m$ is null-homotopic.

5. Let $X = S^1 \times [0, 2\pi]$. Let $f : X \to X$ be the map whose restriction to $S^1 \times \{t\}$ is a rotation of t radians counter-clockwise.

(a) Show that f is homotopic to the identity map.

(b) Is it possible to find a homotopy between f and the identity map that fixes every point in $S^1 \times 0$ and every point in $S^1 \times 1$? Explain.

6. Let Σ_g be the surface of genus g.

(a) Define the Euler characteristic of a CW complex.

(b) Show that for every $g > 2$, Σ_g is a finite cover of Σ_2, and that Σ_2 is not a finite cover of Σ_g.

(c) Show that Σ_4 is a not a finite cover of Σ_3.

7. Let X be a surface of genus two, and let Y be a torus. View X as the union of two one holed tori glued together along their boundaries. Call one such one holed torus A. Let $f : X \to Y$ be the map obtained by contracting A to a point. Write out the long exact sequence in homology associated to the pair (X, A), and calculate all the terms in this sequence. What is the degree of f? Carefully state the theorems you use.

8. Let $X = S^1 \times D^2$ be a circle times a closed disk. Glue two copies of X to each other along their boundaries using the identity map on the boundary of X. Calculate the homology groups of the resulting space. Then calculate the homology groups of $X \times S^2$.