Topology qualifying exam sample

The exam consists of eight questions split into parts. You have four hours to complete it. In order to pass, you must exhibit substantial knowledge and ability in each of the following fields: general topology, homotopy theory and covering spaces, and homology theory.

1. Suppose that X is compact, Y is Hausdorff, and $f : X \to Y$ is continuous. Show that $f(X)$ is compact.

2. Let X be the space $\mathbb{R}$ with the metric topology, let $\sim$ be the relation on $\mathbb{R}$ defined by $x \sim y$ iff $x - y \in \mathbb{Q}$, and let $Y = \mathbb{R}/\sim$ with the quotient topology.

(a) Describe the open sets in Y.
(b) Describe the continuous functions from Y to X.
(c) Show that Y is not Hausdorff.

3.

(a) Define the following terms: regular covering space, deck group, universal cover.
(b) Let $G = (\mathbb{Z}/2\mathbb{Z}) \star (\mathbb{Z}/3\mathbb{Z})$. Explain how you would construct a space X such that $\pi_1(X) = G$, and the universal cover of X is contractible.

4. Let F_3 be the free group of rank 3. How many subgroups of index 2 does it have?

5. Let Γ be a finite group that acts without fixed points on the sphere S^3 and let $X = S^3/\Gamma$. Show that any continuous map from X to a graph is null-homotopic.

6. Show that any continuous map $f : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

7. Let $D = \{z \in \mathbb{C} \| |z| \leq 1\}$. Let X be the space obtained by gluing two copies of D to each other by identifying z in the boundary of the first disk with z^3 in the boudary of the other one.

(a) Define a CW complex structure and a Δ-complex structure on X.
(b) Calculate the cellular homology groups of X.

8. Let $X = \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$. Compute $H_*(X, X - \{(0, 0, 0)\})$, then prove that any homeomorphism $f : X \to X$ must fix the origin.