Satellite Orbital Control

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SATELLITE ORBITAL CONTROL

REGINA EVANS

Abstract. In this project we will be looking at the change in control required to transfer a satellite between two elliptic Keplerian orbits. We will first derive the equations of motion for our satellite and then study the controllability properties of our system. We will introduce a simple feedback controller and prove local asymptotic stability of the target orbit. The goal of this paper is to prove stability using both geometric control theory as well as stabilization methods and thus link the work done by [3] and [8]. Our primary tool to accomplish this will be LaSalle's invariance principle.

1. Introduction

Much study has been done in the field of geometric control applied to systems of aerospace engineering. An important area of study is the transfer of a satellite between elliptic orbits. Since the 1990s, research projects have focused on low electro-ionic propulsion rather than stronger chemical propulsion, taking into account the very low thrust and longer transfer duration involved. There are two primary techniques used in this field of study. The first technique studies orbital transfer using stabilization methods with simple feedback controllers. Early work in this field did not take into account the time of transfer, as in [3], while more recent analysis seeks stabilization feedbacks which achieve time efficient transfer, see [7]. The second technique involves performing orbital transfers while minimizing a cost function. We may seek to minimize the fuel consumption (and maximize the final mass) or attempt to minimize the transfer time. Because orbital transfer using low propulsion can take several months, more attention has been given to the time minimization problem. For more on the time optimal control problem see [8, 9] among others.

This paper will focus on the use of stabilization techniques via a simple feedback controller. We will first present the model for our system, deriving the Kepler equation and constants of motion. We shall see that we can uniquely describe the orbit at all times by the angular momentum and Laplace vectors. We will analyze the controllability of our system using Lie bracket computations and then see how to construct a feedback control using stabilization techniques. Our goal is to prove stability of a Lyapunov-based controller using geometric control techniques and compare this to the proof presented in [3] using stabilization methods.

2. The Model

We will begin by introducing the model for our system and in celestial mechanics this means starting with the central force problem. Throughout this paper we shall analyze the motion of a satellite of mass m as it is attracted to a fixed center O, in our case the Earth center. For our purposes we shall agree that the mass of the satellite is significantly less than that of our central

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body (the Earth) and we will assume that the center of mass of our system is the same as the center of mass of our central body. We will use the laws of planetary motion developed by Johannes Kepler in the early 1600's and their application to orbital mechanics together with the laws of attraction developed by Isaac Newton to describe our system.

Newton's law of gravitation tells us the attractive force governing the motion of a satellite of mass m and our central body of mass M depends only on each objects mass and is inversely proportional to the square of the distance r between them (see [1]). Newton's second law of motion says that this force equals the satellite's mass times its acceleration. Setting the two forces equal we have

$$m\,\ddot{\mathbf{r}} = \frac{M \cdot m}{r^2}\,\frac{-\mathbf{r}}{r}\,G \tag{2.1}$$

Here $\mathbf{r}$ is the position vector from O to the satellite, r its magnitude, and thus $\frac{\mathbf{r}}{r}$ is the radial unit vector. G is the universal gravitational constant which is approximately equal to $6.67428 \times 10^{-11}\, m^3\, kg^{-1}\, s^{-2}$.

We define our configuration space as $\mathbb{R}^3$ minus the origin which we will write as $\mathbb{R}^3_O := \mathbb{R}^3 - \{O\}$ and our tangent space as $T\mathbb{R}^3_O = (\mathbb{R}^3 - \{O\}) \times \mathbb{R}^3$. (See Appendix A for more on the configuration space.) We will begin to analyze the vector functions $\mathbf{r}(\mathrm{t})$ and $\dot{\mathbf{r}}(\mathrm{t})$ which are solutions to equation (2.1). We use $(\mathbf{r},\dot{\mathbf{r}})$ as our coordinates for $T\mathbb{R}^3_O$ with the over-dot as the derivative with respect to time.

2.1. **Kepler Equation and Constants of Motion.** We note from equation (2.1) that the mass of the satellite cancels and thus is irrelevant to this equation of motion. We introduce the positive constant $\mu = GM$ and rewrite equation (2.1) as

$$\ddot{\mathbf{r}} = \frac{-\mu \mathbf{r}}{r^3} \tag{2.2}$$

We shall call this *the Keplerian equation of motion* and refer to solutions of this equation as *Keplerian orbits*. As mentioned earlier we are assuming the mass of the satellite is significantly less than that of our central body and we require that the distance between them remains limited.

We continue by analyzing a few constants of motion which we derive from the Keplerian equation. First we examine the derivative of the vector $\mathbf{r} \times \dot{\mathbf{r}}$, which is $\mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dot{\mathbf{r}}$. Using equation (2.2) we observe that $\mathbf{r} \times \ddot{\mathbf{r}} = -\frac{\mu}{r^3}(\mathbf{r} \times \mathbf{r}) = 0$ and since $\dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$, we see that the derivative of $\mathbf{r} \times \dot{\mathbf{r}} = 0$. Hence the vector $\mathbf{r} \times \dot{\mathbf{r}}$ is our first constant of motion, and we define it to be the *angular momentum*, $\mathbf{L}$, of the satellite. As the position and velocity vectors evolve, the angular momentum vector remains unchanged. This relation is known as the *conservation of angular momentum*. We define the following:

$$\begin{aligned} \mathbf{L}(\mathbf{r},\dot{\mathbf{r}}) &: T\mathbb{R}^3_O \to \mathbb{R}^3 \\ \mathbf{L}(\mathbf{r},\dot{\mathbf{r}}) &= \mathbf{r} \times \dot{\mathbf{r}} \end{aligned} \tag{2.3}$$

It is important to note that $\mathbf{r} \cdot \mathbf{L} = 0$, so when $\mathbf{L} \neq 0$ we see $\mathbf{r}$ is always perpendicular to the fixed vector $\mathbf{L}$. Thus all motion of $\mathbf{r}$ takes place in a fixed plane through O and perpendicular to $\mathbf{L}$. If $\mathbf{L} = 0$, the vector $\frac{\mathbf{r}}{r}$ is constant (as seen in the development of the Laplace vector below), and the motion of $\mathbf{r}$ takes place along a fixed straight line through O. In such a case we say $(\mathbf{r}, \dot{\mathbf{r}})$ is a *degenerate orbit*, and for practical purposes we shall not concern ourselves with this case.

Another important vector which remains constant throughout the motion of the satellite is called *the Laplace vector*, $\mathbf{A}$ (and sometimes called the *eccentric axis*, $\mathbf{e}$, when scaled by μ). To derive the

Laplace vector we begin by analyzing the derivative $\frac{d}{dt}\frac{\mathbf{r}}{r}$. Since $r^2 = \mathbf{r}\cdot\mathbf{r}$, we have $r\dot{r} = \mathbf{r}\cdot\dot{\mathbf{r}}$, and thus

$$\begin{aligned}\frac{d}{dt}\frac{\mathbf{r}}{r} &= \frac{\dot{\mathbf{r}}r - \mathbf{r}\dot{r}}{r^2}\\ &= \frac{1}{r^2}\left[\dot{\mathbf{r}}(\frac{\mathbf{r}\cdot\mathbf{r}}{r}) - \mathbf{r}(\frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r})\right]\\ &= \frac{(\mathbf{r}\times\dot{\mathbf{r}})\times\mathbf{r}}{r^3} = \frac{\mathbf{L}\times\mathbf{r}}{r^3}\end{aligned}$$

We continue by multiplying both sides by $-\mu$ and we apply our Keplerian equation (2.2) to obtain the following

$$\begin{aligned}-\mu\frac{d}{dt}\frac{\mathbf{r}}{r} &= \mathbf{L}\times\frac{-\mu\mathbf{r}}{r^3}\\ \mu\frac{d}{dt}\frac{\mathbf{r}}{r} &= \ddot{\mathbf{r}}\times\mathbf{L}\end{aligned}$$

We pause here to note that if $\mathbf{L} = 0$, then $\frac{d}{dt}\frac{\mathbf{r}}{r} = 0$ and $\frac{\mathbf{r}}{r}$ is constant which is the degenerate case mentioned earlier. We continue and integrate both sides and introduce the constant of integration $\mathbf{e}$ to get

$$\mu(\mathbf{e} + \frac{\mathbf{r}}{r}) = \dot{\mathbf{r}}\times\mathbf{L} \tag{2.4}$$

Thus we define *the Laplace vector*, $\mathbf{A}$, as follows noting that it too remains constant

$$\mathbf{A(r,\dot{r})} : T\mathbb{R}^3_O \to \mathbb{R}^3$$

$$\mathbf{A(r,\dot{r})} = \dot{\mathbf{r}}\times\mathbf{L} - \mu\frac{\mathbf{r}}{r} = \mu\,\mathbf{e} \tag{2.5}$$

Finally we turn to analyze the total energy of our system which also turns out to be constant. The total energy, mE, of a system under Keplerian motion is the sum of its *kinetic energy*, $\frac{1}{2}m\dot{r}^2$ and its *potential energy*, $-\frac{m\mu}{r}$. This relationship is known as *the principle of conservation of energy*, and we shall define the energy, E, of our system as follows

$$E(\mathbf{r},\dot{\mathbf{r}}) : T\mathbb{R}^3_O \to \mathbb{R}$$

$$E(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}\dot{r}^2 - \frac{\mu}{r} \tag{2.6}$$

2.2. **Properties of Constants of Motion.** We shall see that our three constants of motion, *the angular momentum* $\mathbf{L}$, *the Laplace vector* $\mathbf{A}$, and the *energy* E, satisfy the following relationships

$$\mathbf{A}\cdot\mathbf{L} = 0 \tag{2.7}$$

$$A^2 = \mu^2 + 2EL^2 \tag{2.8}$$

Equation (2.7) can easily be seen by recalling $\mathbf{r}\cdot\mathbf{L} = 0$ and thus $\mathbf{A}\cdot\mathbf{L} = (\dot{\mathbf{r}}\times\mathbf{L} - \mu\frac{\mathbf{r}}{r})\cdot\mathbf{L} = 0$. Equation (2.8) can be derived by first squaring equation (2.4) and observing that since $\dot{\mathbf{r}}$ is perpendicular to $\mathbf{L}$ we can write $(\dot{\mathbf{r}}\times\mathbf{L})^2 = \dot{r}^2L^2$ and we have

$$\mu^2\,(\mathbf{e}+\frac{\mathbf{r}}{r})^2 = \dot{r}^2 L^2$$
$$\mu^2\,(e^2+\frac{2}{r}\,\mathbf{e}\cdot\mathbf{r}+1) = \dot{r}^2 L^2$$

Now using equation (2.6) we can replace $\dot{r}^2$ with $2E+\frac{2\mu}{r}$. Also, taking the dot product of $\mathbf{r}$ with both sides of equation (2.4) we observe that $\mu(\mathbf{e}\cdot\mathbf{r}+r) = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L}) = \mathbf{L}\cdot(\mathbf{r}\times\dot{\mathbf{r}}) = \mathbf{L}\cdot\mathbf{L}$. Thus we can replace $\mathbf{e}\cdot\mathbf{r}$ with $\frac{L^2}{\mu}-r$. After applying some algebra we have equation (2.8).

We saw earlier that when $\mathbf{L}=0$, then $(\mathbf{r},\dot{\mathbf{r}})$ is a degenerate orbit and $\mathbf{r}(\mathrm{t})$ moves in a straight line. If $\mathbf{L}\neq 0$ we use the relation $\mu(\mathbf{e}\cdot\mathbf{r}+r) = \mathbf{L}\cdot\mathbf{L}$ to examine two cases. If $\mathbf{A}(=\mathbf{e}\mu) = 0$ then $r=\frac{L^2}{\mu}$ which is constant. Thus the motion $\mathbf{r}(\mathrm{t})$ is circular. Alternatively, if $\mathbf{A}(=\mathbf{e}\mu)\neq 0$ we examine the plane of motion of the vector $\mathbf{r}$ which lies perpendicular to $\mathbf{L}$. We introduce the vector $\mathbf{e}$ as shown in Figure 1. We denote the angle from the x-axis to $\mathbf{e}$ by ω and represent the position of the satellite by (r,θ). Using the identity $\mathbf{e}\cdot\mathbf{r} = e\,r\cos(\theta-\omega)$ we rewrite the relation $\mu(\mathbf{e}\cdot\mathbf{r}+r)=\mathbf{L}\cdot\mathbf{L}$ as

$$r = \frac{\frac{L^2}{\mu}}{1+e\,\cos(\theta-\omega)} = \frac{L^2}{\mu+A\,\cos(\theta-\omega)} = e\left(\frac{L^2}{A}-r\cos(\theta-\omega)\right) \tag{2.9}$$

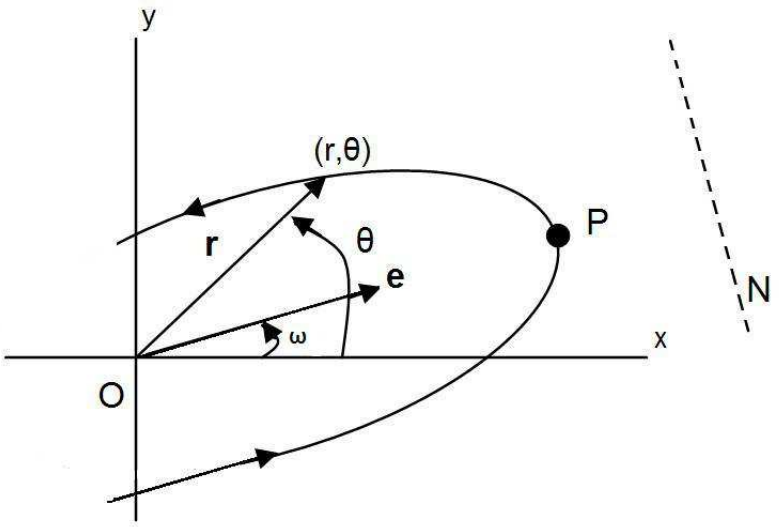


FIGURE 1. Kepler's first law

Consider the line N drawn at a distance $\frac{L^2}{A}$ from O and perpendicular to $\mathbf{e}$ as shown in Figure1. Equation (2.9) says that the distance of the satellite from O is e times its distance from N. This tells us that our satellite moves in a conic section of eccentricity e with a focus at O. This is known as *Kepler's First Law*. We see from Equation (2.9) that r is smallest when $\theta-\omega=0$. Hence the vector $\mathbf{e}$ has length equal to the eccentricity and points to the position P where the satellite orbits closest to the focus. Astronomers refer to the point P as the *periapsis*. We shall call the angle

ω the *arguement of periapsis* and the angle $\theta - \omega$ the *true anomaly*. Various names are given to the periapsis, according the source of attraction O. In our case the source O is the Earth and P is called the *perigee*. Analyzing equations (2.8) and (2.9) we observe that $\mathbf{r}(\mathrm{t})$ traces out the following conic sections depending on the energy of the system:

Conic Section	Energy	Laplace Vector
ellipse	$E < 0$	$A < \mu$ (or $e < 1$)
parabola	$E = 0$	$A = \mu$ (or $e = 1$)
hyperbola	$E > 0$	$A > \mu$ (or $e > 1$)

2.3. **Elliptic Domain.** As mentioned earlier, we will exclude degenerate orbits from our discussion, and we shall now focus on elliptic orbits in negative energy systems. Here we introduce the *elliptic domain* and demonstrate that we can uniquely describe every (oriented) elliptic Keplerian orbit by the pair (**L**,**A**). This result will be very useful in our later analysis when we define our Lyapunov controller.

We define the set $\Sigma_e = \{(\mathbf{r}, \dot{\mathbf{r}}) \in T\mathbb{R}^3_O \,|\, \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}) \neq 0, E(\mathbf{r}, \dot{\mathbf{r}}) < 0\}$ and call it the *elliptic domain*. We see that the elliptic domain is filled by elliptic Keplerian orbits. Define the map $\pi : T\mathbb{R}^3_O \to \mathbb{R}^3 \times \mathbb{R}^3$ by $\pi(\mathbf{r}, \dot{\mathbf{r}}) = (\mathbf{L}, \mathbf{A})$ and define the set $D = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, \mathbf{x} \cdot \mathbf{y} = 0,\, \mathbf{x} \neq 0,\, y < \mu\}$. Observe from equations (2.7) and (2.8) that

$$\pi(\Sigma_e) \subset D \text{ and } \pi(T\mathbb{R}^3_O - \Sigma_e) \cap D = \varnothing \tag{2.10}$$

which implies $\pi^{-1}(D) = \Sigma_e$. Now we would like to show that $\pi(\Sigma_e) = D$. For any $(\mathbf{x}, \mathbf{y}) \in D$ we take

$$(\mathbf{r}, \dot{\mathbf{r}}) = \begin{cases} \left(\frac{x^2}{y(y+\mu)}\,\mathbf{y},\ \ \frac{y+\mu}{x^2 y}\,\mathbf{x} \times \mathbf{y}\right) & \text{if } \mathbf{y} \neq 0 \\ \\ \left(\frac{x^2}{\mu^2}\,\mathbf{p} \times \mathbf{x},\ \ \mathbf{p}\right) & \text{if } \mathbf{y} = 0 \end{cases}$$

where $\mathbf{p}$ is a vector such that $\mathbf{p} \cdot \mathbf{x} = 0$ and $p^2 = \frac{y^2 - \mu^2}{x^2}$. We calculate E($\mathbf{r}$,$\dot{\mathbf{r}}$) and $\mathbf{L}$($\mathbf{r}$,$\dot{\mathbf{r}}$) as follows:

$$\text{if } \mathbf{y} \neq 0, \text{ then } E(\mathbf{r}, \dot{\mathbf{r}}) = \tfrac{y^2 - \mu^2}{2x^2} < 0 \text{ and } \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}) = \mathbf{x} \neq 0, \text{ and}$$

$$\text{if } \mathbf{y} = 0, \text{ then } E(\mathbf{r}, \dot{\mathbf{r}}) = \tfrac{-3\mu^2}{2x^2} < 0 \text{ and } \mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}) = \mathbf{x} \neq 0.$$

Thus $(\mathbf{r}, \dot{\mathbf{r}}) \in \Sigma_e$. Similarly we calculate $\pi(\mathbf{r}, \dot{\mathbf{r}}) = (\mathbf{x}, \mathbf{y})$ and conclude D $\subset \pi(\Sigma_e)$. Combining this result with line (2.10) we have $\pi(\Sigma_e) = D$.

To prove uniqueness we first observe that since $\mathbf{L}$ and $\mathbf{A}$ are constants, and since $\pi(\Sigma_e) = D$ and $\pi^{-1}(D) = \Sigma_e$, then for each $(\mathbf{x}, \mathbf{y}) \in D$, $\pi^{-1}(\mathbf{x}, \mathbf{y})$ consists of a union of elliptic Keplerian orbits. Now given a pair $(\mathbf{L}, \mathbf{A}) \in D$, we take any elliptic Keplerian orbit ($\mathbf{r}$(t), $\dot{\mathbf{r}}$(t)) contained in $\pi^{-1}(\mathbf{L}, \mathbf{A}) \subset T\mathbb{R}^3_O$. We know that the orbit ($\mathbf{r}$(t), $\dot{\mathbf{r}}$(t)) occurs in the set $\Pi \times \Pi$ where $\Pi \subset \mathbb{R}^3$ is the plane through the origin O and perpendicular to $\mathbf{L}$ (because both $\mathbf{r}$(t) and $\dot{\mathbf{r}}$(t) are perpendicular to $\mathbf{L}$). We recall from above that the polar equation (r, θ) of the ellipse traced out by $\mathbf{r}$(t) on Π is

$$r = \frac{L^2}{\mu + A\cos(\theta - \omega)}$$

where ω is the argument of periapsis. From equations (2.3) and (2.4), we calculate the tangent vector $\dot{\mathbf{r}}$ at $\mathbf{r}$ as:

$$\dot{\mathbf{r}} = \frac{\mathbf{L}}{L^2} \times \left(\mathbf{A} + \frac{\mu \mathbf{r}}{r} \right).$$

Thus it follows that for $(\mathbf{L}, \mathbf{A}) \in D$, $\pi^{-1}(\mathbf{L}, \mathbf{A})$ consists of a unique oriented elliptic Keplerian orbit. We conclude that

(i) Σ_e is the union of all elliptic Keplarian orbits,

(ii) $\pi(\Sigma_e) = D$ and $\Sigma_e = \pi^{-1}(D)$, and

(iii) For each $(\mathbf{x}, \mathbf{y}) \in D$, the fiber $\pi^{-1}(\mathbf{x}, \mathbf{y})$ consists of a unique oriented elliptic Keplerian orbit.

3. Control Force

Now that we understand the motion of a satellite under the Keplarian equation, we will introduce a control force which will allow us to transfer a satellite between elliptic orbits. In this section we shall introduce the controlled Kepler equation and study the controllability properties of the system.

3.1. **Controlled Kepler equation.** Our primary focus from this point forward will be the control system

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3} \mathbf{r} + \mathbf{F} \tag{3.1}$$

which we shall call *the controlled Kepler equation (with constant mass)*. Note again that $\mathbf{r}$ is the position vector from our fixed center O to the satellite measured in the fixed frame I, J, K, and $\mathbf{F}$ is the control force, or thrust, which we shall use to steer the satellite from one elliptical orbit to another. In general the control force can be designed in two basically different ways. In *open loop* form the control law is a function of time: $t \mapsto \mathbf{F}(t)$, while in *closed loop*, or *feedback* form, the control law is a function of the state, in our case $(\mathbf{r}, \dot{\mathbf{r}}) \mapsto \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$. Open loop control is easier to implement because the only information needed is a clock to measure time. On the other hand, a closed loop control requires constant measurement of the state of the system. An advantage of closed loop control however is that by measuring the behavior of the system, the controls can react to the difference between what the system is supposed to be doing and what it is actually doing. As we shall see in Section 4, this paper shall utilize a closed loop feedback controller to achieve asymptotically stable local orbit transfer.

We pause here to mention that the general form of the controlled Kepler equation uses a control force $\frac{\mathbf{F}}{m}$ where m is the mass of the satellite which evolves over time. Any analysis of orbital transfer which seeks to minimize a cost function must use the general form of the controlled Kepler equation. Such work often seeks to minimize fuel consumption or transfer time. This paper however will deal only with the constant mass model, which is a simplified version sufficient for our geometric analysis. Note though that the mass equation should be considered when doing any numerical computations.

We now introduce the vector fields $\mathbf{F}_i = \frac{\partial}{\partial \dot{r}_i}$, for i=1,2,3 identified to I, J, K respectively, and see that the thrust can be decomposed as $\mathbf{F} = \Sigma_{i=1}^{3} u_i \mathbf{F}_i$ where the u_i's are the Cartesian components of the control. We can obtain a more physical decomposition however if we decompose the thrust into a moving frame attached to the satellite. Two common frames used with the controlled Kepler

equation (in the case of nondegenerate orbits, i.e. when $\mathbf{L} = \mathbf{r} \times \dot{\mathbf{r}} \neq 0$) are the *tangential-normal* and the *radial-orthoradial* frames. We shall focus on the *tangential-normal* frame in which $\mathbf{F} = u_t \mathbf{F}_t + u_n \mathbf{F}_n + u_l \mathbf{F}_l$ where

$$\mathbf{F}_t = \frac{\dot{\mathbf{r}}}{\dot{r}} \frac{\partial}{\partial \dot{r}}, \ \mathbf{F}_l = \frac{\mathbf{r} \times \dot{\mathbf{r}}}{r \times \dot{r}} \frac{\partial}{\partial \dot{r}}, \text{ and } \mathbf{F}_n = \mathbf{F}_l \times \mathbf{F}_t.$$

Observe that by setting the control force $u_l = 0$ we have a 2D-problem where the motion of our satellite is restricted to the osculating plane spanned by the original position and velocity vectors $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$.

3.2. **Lie Algebraic Structure.** We would like to study the effects of our control force in each direction. This will allow us to understand the action of each physical actuator. To do this we will restrict the control to one direction at a time and study the single-input system $\dot{\mathbf{x}} = \mathbf{F}_0 + u\mathbf{F}_1$. For notational clarity, in this section we will write $\mathbf{v} = \dot{\mathbf{r}}$, $|\mathbf{r}| = r$, and $|\mathbf{v}| = v$. Thus $\mathbf{x} = (\mathbf{r}, \mathbf{v}) \in T\mathbb{R}^3_O \subset \mathbb{R}^6$, and $\mathbf{F}_0 = \mathbf{v} \frac{\partial}{\partial r} - \frac{\mu}{r^3} \mathbf{r} \frac{\partial}{\partial v}$. To perform this geometric analysis we must investigate the Lie structure of the systems. We will first make computations of the Lie algebra $\text{Lie}_x(\mathbf{F}_0, \mathbf{F}_1)$ and later deduce controllability properties of the system. For more on the Lie bracket and Lie algebra see Appendix B.

3.2.1. *Tangential Direction.* We first study the thrust oriented along $\mathbf{F}_t$ by setting $u_n = u_l = 0$, and we see that we are left with a 2D-system defined by our initial position and velocity vectors. We have $\dot{\mathbf{x}} = \mathbf{F}_0 + u\mathbf{F}_t$ with $\mathbf{F}_t = \frac{\mathbf{v}}{v} \frac{\partial}{\partial v}$. We compute the Lie brackets and get

$$\begin{aligned}
[\mathbf{F}_0, \mathbf{F}_t] &= -\frac{1}{v}\mathbf{F}_0 - \frac{\mu(\mathbf{r} \cdot \mathbf{v})}{r^3 v^2}\mathbf{F}_t + \frac{2\mu}{r^3 v^3} (\mathbf{r} \times \mathbf{v}) \times \mathbf{v} \frac{\partial}{\partial v} \\
[\mathbf{F}_t, [\mathbf{F}_0, \mathbf{F}_t]] &= -\frac{2\mu(\mathbf{q} \times \mathbf{v}) \times \mathbf{v}}{r^3 v^4} \frac{\partial}{\partial v} = -\frac{1}{v^2}\mathbf{F}_0 - \frac{\mu(\mathbf{r} \cdot \mathbf{v})}{r^3 v^3}\mathbf{F}_t - \frac{1}{v}[\mathbf{F}_0, \mathbf{F}_t] \\
[\mathbf{F}_0, [\mathbf{F}_0, \mathbf{F}_t]] &= -\frac{2\mu}{r^3 v^3} (\mathbf{r} \times \mathbf{v}) \times \mathbf{v} \frac{\partial}{\partial r} + a_1 \mathbf{F}_0 + a_2 \mathbf{F}_t + a_3 [\mathbf{F}_0, \mathbf{F}_t]
\end{aligned}$$

where

$$\begin{aligned}
a_1 &= \frac{\mu(\mathbf{r} \cdot \mathbf{v})}{r^3 v^3} - \frac{3(\mathbf{r} \cdot \mathbf{v})}{r^2 v} \\
a_2 &= -\frac{\mu}{r^3} + \frac{\mu^2((\mathbf{r} \cdot \mathbf{v})^2 - |\mathbf{r} \times \mathbf{v}|^2)}{r^6 v^4} \\
a_3 &= \frac{\mu(\mathbf{r} \cdot \mathbf{v})}{r^3 v^2} - \frac{3(\mathbf{r} \cdot \mathbf{v})}{r^2}
\end{aligned}$$

We see that the vector fields $\mathbf{F}_0, \mathbf{F}_t, [\mathbf{F}_0, \mathbf{F}_t]$, and $[\mathbf{F}_0, [\mathbf{F}_0, \mathbf{F}_t]]$ are linearly independent and form a frame. Thus, the rank of the Lie algebra $\text{Lie}_x(\{\mathbf{F}_0, \mathbf{F}_t\})$ is four.

3.2.2. *Normal Direction.* We now study the thrust oriented along $\mathbf{F}_n$ by setting $u_t = u_l = 0$. We again have a 2D-system where $\dot{\mathbf{x}} = \mathbf{F}_0 + u\mathbf{F}_n$ with $\mathbf{F}_n = \frac{(\mathbf{r} \times \mathbf{v}) \times \mathbf{v}}{|(\mathbf{r} \times \mathbf{v}) \times \mathbf{v}|} \frac{\partial}{\partial v}$. We compute the Lie brackets

and get

$$\begin{aligned}
[\mathbf{F}_0, \mathbf{F}_n] &= -\frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|v}\,\frac{\partial}{\partial r} - \frac{\mu\,|\mathbf{r}\times\mathbf{v}|}{r^3\,v^3}\mathbf{v}\,\frac{\partial}{\partial v} \\
[\mathbf{F}_n, [\mathbf{F}_0, \mathbf{F}_n]] &= \frac{1}{v^2}\mathbf{F}_0 - c_1\,\mathbf{F}_n \\
[\mathbf{F}_0, [\mathbf{F}_0, \mathbf{F}_n]] &= c_1\,\mathbf{F}_0 + c_2\,\mathbf{F}_n
\end{aligned}$$

where

$$\begin{aligned}
c_1 &= \frac{2\mu\,|\mathbf{r}\times\mathbf{v}|}{r^3\,v^3} \\
c_2 &= -\frac{3\mu^2\,|\mathbf{r}\times\mathbf{v}|^2}{r^6\,v^4} - \frac{3\mu(\mathbf{r}\cdot\mathbf{v})^2 - 2r^2\,v^2}{r^5\,v^2}
\end{aligned}$$

We see that the brackets of length three are contained in the span of $(\{\mathbf{F}_0, \mathbf{F}_n\})$. Thus we conclude that the vector fields $\mathbf{F}_0, \mathbf{F}_n, [\mathbf{F}_0, \mathbf{F}_n]$ form a frame and the rank of the Lie algebra $\mathrm{Lie}_x(\{\mathbf{F}_0, \mathbf{F}_n\})$ is three.

3.2.3. *Momentum Direction.* Finally we study the thrust oriented along $\mathbf{F}_l$ by setting $u_t = u_n = 0$, and we have the system $\dot{\mathbf{x}} = \mathbf{F}_0 + u\mathbf{F}_l$ with $\mathbf{F}_l = \frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|}\frac{\partial}{\partial v}$. We compute the Lie brackets and get

$$\begin{aligned}
[\mathbf{F}_0, \mathbf{F}_l] &= -\frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|}\,\frac{\partial}{\partial r} \\
[\mathbf{F}_l, [\mathbf{F}_0, \mathbf{F}_l]] &= \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{|\mathbf{r}\times\mathbf{v}|^2}\,\frac{\partial}{\partial r} + \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|^2}\,\frac{\partial}{\partial v} \\
[\mathbf{F}_0, [\mathbf{F}_0, \mathbf{F}_l]] &= -\frac{\mu}{r^3}\,\mathbf{F}_l \\
[\mathbf{F}_l, [\mathbf{F}_l, [\mathbf{F}_0, \mathbf{F}_l]]] &= -\frac{r^2}{|\mathbf{r}\times\mathbf{v}|^2}[\mathbf{F}_0, \mathbf{F}_l] + \frac{\mathbf{r}\cdot\mathbf{v}}{|\mathbf{r}\times\mathbf{v}|^2}\mathbf{F}_l \\
[\mathbf{F}_0, [\mathbf{F}_l, [\mathbf{F}_0, \mathbf{F}_l]]] &= 0
\end{aligned}$$

We see that the vector fields $\mathbf{F}_0, \mathbf{F}_l$, and $[\mathbf{F}_0, \mathbf{F}_l]$ are linearly independent. If the Laplace vector $\mathbf{A} \neq 0$ then $\mathbf{F}_0, \mathbf{F}_l, [\mathbf{F}_0, \mathbf{F}_l]$ and $[\mathbf{F}_l, [\mathbf{F}_0, \mathbf{F}_l]]$ form a frame of the Lie algebra $\mathrm{Lie}_x(\{\mathbf{F}_0, \mathbf{F}_l\})$ with dimension four. Recall that the Laplace vector $\mathbf{A} = \mathbf{v}\times(\mathbf{r}\times\mathbf{v}) - \frac{\mu}{r}\mathbf{r}$, and $\mathbf{A} = 0$ corresponds to circular orbits in which $\mathbf{r}\cdot\mathbf{v} = 0,\ \mu = v^2 r$. Thus if $\mathbf{A} = 0$, then r and v are constant and we have $[\mathbf{F}_l, [\mathbf{F}_0, \mathbf{F}_l]] = \frac{1}{v^2}\mathbf{F}_0$ which tells us the dimension of the Lie algebra $\mathrm{Lie}_x(\{\mathbf{F}_0, \mathbf{F}_l\})$ is three.

3.3. **Orbital Elements.** We described earlier that each elliptic Keplerian orbit can be uniquely described by the angular momentum and Laplace vectors (**L**,**A**). We will now introduced a more detailed representation provided by the orbital elements with the hopes of providing further insight into our system. With the I, J, K reference frame, we will identify the plane (I,J) with the Earth equatorial plane. We can then represent each point $(\mathbf{r}, \mathbf{v})$ of the elliptic domain by the following geometric parameters of the osculating orbit:

- Ω : angle, or longitude, of the ascending node; The oriented ellipse cuts the (I,J) plane in two opposing points which defines the *line of nodes*.

- ω : argument of the periapsis; This is the angle between the axis of the ascending node and the axis of the periapsis.
- i : inclination of the osculating plane;
- a : semi-major axis of the ellipse;
- e : eccentricity;
- l : cumulated longitude; We have that $l = \Omega + \omega + v$ where v is the true anomaly.

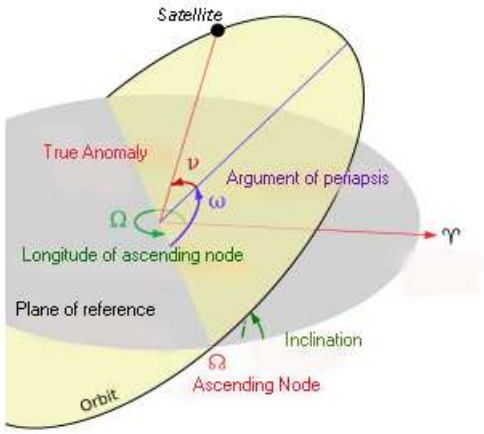


FIGURE 2. Orbital elements

If the eccentricity vector $\mathbf{e}$ is collinear to the Laplace vector $\mathbf{A}$, we let $\tilde{\omega}$ be the angle between I and $\mathbf{e}$. We define $e_1 = e\cos\tilde{\omega}$, and $e_2 = e\sin\tilde{\omega}$. Further, we represent the line of nodes contained in the (I,J) plane by $h_1 = \tan\frac{i}{2}\cos\Omega$, $h_2 = \tan\frac{i}{2}\sin\Omega$. These systems of equations are commonly referred to as *Gauss equations*. The following results are standard in orbital mechanics. Using the coordinates $\mathbf{x} = (a, e_1, e_2, h_1, h_2, l)$ we decompose the thrust into the tangential-normal frame $\dot{\mathbf{x}} = \mathbf{F}_0 + u_t\mathbf{F}_t + u_n\mathbf{F}_n + u_l\mathbf{F}_l$. In these coordinates we have

$$\mathbf{F}_0 = \sqrt{\frac{\mu}{P}}\frac{W^2}{P}\frac{\partial}{\partial l},$$

$$\mathbf{F}_t = \frac{1}{W}\sqrt{\frac{P}{\mu}}\left(\frac{2WP|\eta|}{(1-\mathrm{e}^2)^2}\frac{\partial}{\partial a} + \frac{2W\eta_1}{|\eta|}\frac{\partial}{\partial e_1} + \frac{2W\eta_2}{|\eta|}\frac{\partial}{\partial e_2}\right),$$

$$\mathbf{F}_n = \frac{1}{W}\sqrt{\frac{P}{\mu}}\left(\frac{D\eta_1 - W\eta_2}{|\eta|}\frac{\partial}{\partial e_1} + \frac{W\eta_1 - D\eta_2}{|\eta|}\frac{\partial}{\partial e_2}\right),$$

$$\mathbf{F}_l = \frac{1}{W}\sqrt{\frac{P}{\mu}}\left(-Ze_2\frac{\partial}{\partial e_1} + Ze_1\frac{\partial}{\partial e_2} + \frac{C\cos l}{2}\frac{\partial}{\partial h_1} + \frac{C\sin l}{2}\frac{\partial}{\partial h_2} + Z\frac{\partial}{\partial l}\right),$$

where

$$\begin{aligned} P &= a(1-e^2)\,, \\ \eta &= (\eta_1,\eta_2) = (e_1+\cos l, e_2+\sin l)\,, \\ W &= 1+e_1\,\cos l + e_2\,\sin l\,, \\ D &= e_1\,\sin l - e_2\,\cos l\,, \\ C &= 1+h^2, \\ Z &= h_1\sin l - h_2\cos l. \end{aligned}$$

We see here that *coplanar transfer* corresponds to the case where the osculating plane is kept fixed, and hence $u_l = 0$ giving us a 2D-system. The interested reader will also note that $|e| = |h| = 0$ corresponds to the geostationary orbit, which is the geosynchronous orbit directly above the Earth's equator (0 latitude), with a period equal to the Earth's rotational period.

3.4. **Controlability Results.** We now combine our Lie bracket computations with the decomposed system in Gauss coordinates to gain some understanding of the controllability of our system. The following controllability result is necessary for our analysis.

Theorem 1- Let M be a connected manifold and consider the smooth system $\dot{\mathrm{x}}(t) = F_0(x(t)) + \Sigma_{i=1}^n u_i(t)F_i(x(t))$, where the controller u_i takes values in $\{-\varepsilon,\varepsilon\}$, $\varepsilon > 0$ for $i = 1,...,n$. If

- the dimension of the Lie algebra $\mathrm{Lie}_x(\{F_0, F_1, .., F_n\})$ equals the dimension of M for every $x \in M$, and
- the vector field F_0 is periodic (and hence Poisson stable),

then the system is controllable on M. (For further exposition and proof, see [10].)

We restrict our attention to the elliptic domain and apply Theorem 1 as we examine the decomposed thrust in each direction. We can do this because the free motion equation $\dot{x} = \mathbf{F}_0(x)$ gives us a periodic elliptic orbit as was seen in our study of the Keplerian equation. For $\mathbf{x} = (\mathbf{r},\mathbf{v}) \in T\mathbb{R}^3_O \subset \mathbb{R}^6$ and $\mathbf{r}\times\mathbf{v} \neq 0$ (i.e. non-degenerate orbits) we observe the following. In the tangential direction we had a 2D-system with the rank of $\mathrm{Lie}_x(\{\mathbf{F}_0,\mathbf{F}_t\})$ being four. Observe that in a 2D-system, $\mathbf{x} \in (\mathbb{R}^2 - \{O\}) \times \mathbb{R}^2$ which is of dimension four. We see with the Gauss coordinates that $\mathbf{F}_0$ and $\mathbf{F}_t$ together give us full control in the 2D-elliptic domain corresponding to coplanar transfer. In the normal direction we also had a 2D-system, but the rank of $\mathrm{Lie}_x(\{\mathbf{F}_0,\mathbf{F}_n\})$ was three. Here we see from the Gauss coordinates that $\mathbf{F}_0$ and $\mathbf{F}_n$ alone do not allow us to control a, the semi-major axis of the ellipse. The orbit is the intersection of the 2D-elliptic domain and osculating plane $\{a = a(0)\}$. Finally we examine the momentum direction where we had the dimension of $\mathrm{Lie}_x\{\mathbf{F}_0,\mathbf{F}_l\})$ equal to four in the case of non-circular orbits (and three in the case of circular orbits). We observe from our system in Gauss coordinates that using $\mathbf{F}_0$ and $\mathbf{F}_l$ alone we cannot control either the semi-major axis a or the magnitude of the eccentricity $|e|$. The orbit is given by $a = a(0)$ and $|e| = |e(0)|$. We conclude that no single thruster alone gives us controllability of the system.

With this new understanding of the individual thrusters we now examine our system with full control. Examining the Lie bracket calculations above we see that the vector fields $\mathbf{F}_0$, $\mathbf{F}_t$, $\mathbf{F}_n$, $\mathbf{F}_l$, $[\mathbf{F}_0,\mathbf{F}_l]$, and $[\mathbf{F}_0,\mathbf{F}_n]$ are linearly independent and form a frame. The dimension of the Lie algebra $\mathrm{Lie}_x(\{\mathbf{F}_0,\mathbf{F}_t,\mathbf{F}_n,\mathbf{F}_l\})$ is six which is the dimension of our tangent space, and we say the vectors

$(\mathbf{F}_0, \mathbf{F}_t, \mathbf{F}_n, \mathbf{F}_l)$ satisfy the ad-condition. Thus for the system restricted to the elliptic domain with full control, we apply Theorem 1 and conclude that every point of the orbit is accessible and our system is controllable.

4. Stabilization

Thus far we have done a lot of work to describe our control system and examine its properties. However, the goal of this paper is to design a controller for orbital transfer between arbitrary elliptic Keplerian orbits and to prove stability. We turn now to examine the Lyapunov-based controller designed in [3] and present their proof establishing stability. We begin by reviewing some of the basics of Lyapunov stability theory. We will then introduce a metric on $\mathbb{R}^3 \times \mathbb{R}^3$ which we will use to define a Lyapunov function and create our control force F. Finally we present LaSalle's invariance principle and use it to prove asymptotic stability of our closed-loop feedback controller.

4.1. **Lyapunov stability theory.** We begin with a brief review of Lyapunov stability. For further detail the reader may consult [4]. Consider the autonomous, or time-invariant, dynamic system described by

$$\dot{x} = f(x) \ , \quad x(0) = x_0 \tag{4.1}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $\bar{x} \in D$ is an *equilibrium point* of equation (4.1), which means $f(\bar{x}) = 0$. Without loss of generality, we will assume that $\bar{x} = 0$ since any equilibrium point may be shifted to the origin by a change of variables. Thus we shall assume $f(x)$ satisfies $f(0) = 0$, and we study the stability of the origin $x = 0$.

We say that the equilibrium point $x = 0$ is *Lyapunov stable*, or simply *stable*, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x_0| < \delta$ then for every $t \geq 0$ we have $|x(t)| < \epsilon$. In addition, we say the equilibrium point is *asymptotically stable* if it is stable and δ can be chosen such that if $|x_0| < \delta$, then $lim_{t\to\infty}\, x(t) = 0$. An equilibrium point is *unstable* if it is not stable. Thus Lyapunov stability means that solutions starting close to the equilibrium point (within a δ ball) remain close to the equilibrium point (within an ϵ ball) forever. Asymptotic stability means that solutions starting close to the equilibrium point remain close and eventually converge to that point.

Because we are dealing with satellite orbits, we are interested in the stability of a periodic orbit. To get there we will need to extend the notion of Lyapunov stability from the stability of an equilibrium point to the stability of an invariant set. Let $x(t)$ be a solution of equation (4.1). We say a set M is an *invariant set* with respect to equation (4.1) if $x(0) \in M \Rightarrow x(t) \in M, \forall\, t \in \mathbb{R}$. This means that if a solution belongs to M at some point in time, then it belongs to M for all future and past time. We say a set M is a *positively invariant set* if $x(0) \in M \Rightarrow x(t) \in M, \forall\, t \geq 0$. Thus if a solution starts in M, then it remains in M for all future time. Now for $M \subset D$ an invariant set of equation (4.1) we define an ϵ-neighbrohood of M by $U_\epsilon = \{x \in \mathbb{R}^n \,|\, \text{dist}(x, M) < \epsilon\}$ where $\text{dist}(x, M)$ is the minimum distance from x to a point in M; that is $\text{dist}(x, M) = \inf_{y\in M}\|x - y\|$.

We say that the invariant set M is *stable* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x_0 \in U_\delta$ then for every $t \geq 0$ we have $x(t) \in U_\epsilon$. In addition, we say the invariant set M is *asymptotically stable* if it is stable and δ can be chosen such that if $x_0 \in U_\delta$ then $lim_{t\to\infty}\, \text{dist}(x(t), M) = 0$. We now apply this definition to the case where the invariant set M is the closed orbit associated with our periodic solution. Let $u(t)$ be a nontrivial periodic solution to equation (4.1) with period T.

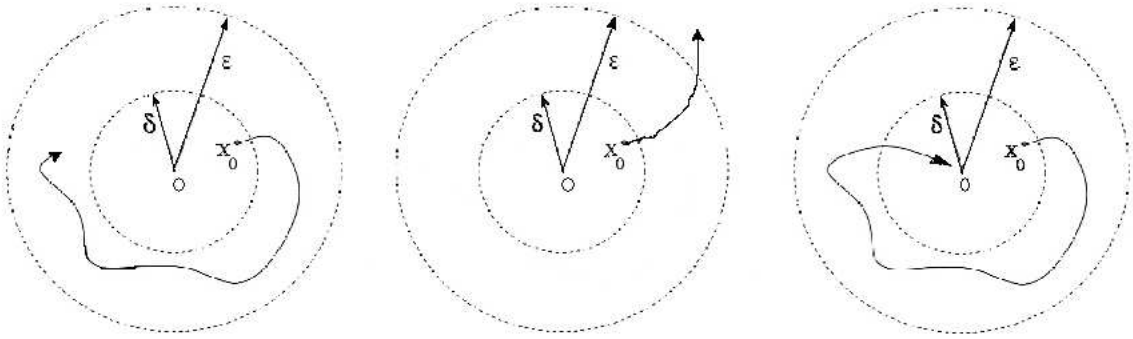


FIGURE 3. An interpretation of stability, instability, and asymptotic stability

We let γ be the closed orbit defined by $\gamma = \{x \in \mathbb{R}^n \,|\, x = u(t), 0 \leq t \leq T\}$. The periodic orbit γ is the image of $u(t)$ and is an invariant set whose stability properties are as defined above.

We finally introduce a bit more terminology and define a Lyapunov function. We say a function $V(x)$ is *positive definite* if $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$. If $V(x)$ satisfies the weaker condition $V(x) \geq 0$ for $x \neq 0$, then we say it is *positive semidefinite*. A function $V(x)$ is *negative definite* or *negative semidefinite* if $-V(x)$ is positive definite or positive semidefinite, respectively. Now we let $x = 0$ be an equilibrium point for equation (4.1), and let $V : D \to \mathbb{R}$ be a continuously differentiable function on a neighborhood D of $x = 0$. We say V is a *Lyapunov function* if $V(0) = 0, V(x) > 0$ in $D \setminus \{0\}$ and $\dot{V}(x) \leq 0$ in $D \setminus \{0\}$. That is, a Lyapunov function $V(x)$ is continuously differentiable, locally positive definite, and its derivative $\dot{V}(x)$ is locally negative semidefinite.

4.2. **Lyapunov-based Controller.** Now that we have a basic notion of stability, our goal in this section is to find a Lyapunov function which gives a feedback controller such that the target elliptic orbit becomes locally asymptotically stable. We begin by defining the following metric d on $\mathbb{R}^3 \times \mathbb{R}^3$:

$$d((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)) = \sqrt{\frac{1}{2}\,|\mathbf{x}_1 - \mathbf{x}_2|^2 + \frac{1}{2}|\mathbf{y}_1 - \mathbf{y}_2|^2}$$

for $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $|\cdot|$ the usual Euclidean norm on $\mathbb{R}^3$. Let $(\mathbf{L}_T, \mathbf{A}_T) \in D$ be the angular momentum and Laplace vector of the target elliptic orbit. For notational simplicity we suppress the dependence of $\mathbf{L}$ and $\mathbf{A}$ on $(\mathbf{r},\dot{\mathbf{r}})$ going forward. We define the Lyapunov function V on $T\mathbb{R}^3_0$ so that $V(\mathbf{r}, \dot{\mathbf{r}})$ is the square of the distance between $(\mathbf{L}, \mathbf{A})$ and the target pair $(\mathbf{L}_T, \mathbf{A}_T)$ in our metric d. Thus

$$\begin{aligned} V(\mathbf{r}, \dot{\mathbf{r}}) &: T\mathbb{R}^3_0 \to \mathbb{R} \\ V(\mathbf{r}, \dot{\mathbf{r}}) &= [d((\mathbf{L}, \mathbf{A}), (\mathbf{L}_T, \mathbf{A}_T))]^2 \\ &= \frac{1}{2}\,|\mathbf{L} - \mathbf{L}_T|^2 + \frac{1}{2}|\mathbf{A} - \mathbf{A}_T|^2 \end{aligned} \tag{4.2}$$

We want to use the Lyapunov function V to create a controller whose direction maximally reduces this distance at each moment. To do this recall that the equation of motion with a control force $\mathbf{F}$ was given by

$$\ddot{\mathbf{r}} = -\mu\frac{\mathbf{r}}{r^3} + \mathbf{F}. \tag{4.2}$$

We see that our system projects in the coordinates $\mathbf{L}$ and $\mathbf{A}$ into

$$\begin{aligned}\frac{d}{dt}\mathbf{L} &= \mathbf{r}\times\mathbf{F}\\ \frac{d}{dt}\mathbf{A} &= (\mathbf{F}\times\mathbf{L}) + \dot{\mathbf{r}}\times(\mathbf{r}\times\mathbf{F})\end{aligned}$$

We write the vectors $\Delta\mathbf{L} = \mathbf{L} - \mathbf{L}_T$ and $\Delta\mathbf{A} = \mathbf{A} - \mathbf{A}_T$, and calculate $\frac{d}{dt}V(\mathbf{r},\dot{\mathbf{r}}) = \mathbf{F}\cdot\mathbf{W}$ where $\mathbf{W} = \Delta\mathbf{L}\times\mathbf{r} + \mathbf{L}\times\Delta\mathbf{A} + (\Delta\mathbf{A}\times\dot{\mathbf{r}})\times\mathbf{r}$. As we shall see in the next section, LaSalle's theorem tells us that we want to choose a thrust $\mathbf{F}$ so that $\frac{d}{dt}V(\mathbf{r},\dot{\mathbf{r}}) \le 0$ along the trajectories. If we choose the controller $\mathbf{F}$ as follows

$$\mathbf{F}(\mathbf{r},\dot{\mathbf{r}};\mathbf{L}_T,\mathbf{A}_T) = -f(\mathbf{r},\dot{\mathbf{r}})\ \mathbf{W} \tag{4.4}$$

with an arbitrary $f(\mathbf{r},\dot{\mathbf{r}}) > 0$, we see that as desired

$$\frac{d}{dt}V(\mathbf{r},\dot{\mathbf{r}}) = -f(\mathbf{r},\dot{\mathbf{r}})\ W^2 \le 0. \tag{4.5}$$

4.3. **LaSalle's Invariance Principle.** Here we state LaSalle's invariance principle and apply it to our closed loop feedback controller to establish stability.

Theorem 2 - (LaSalle's Invariance Principle) Let Ω be a compact (closed and bounded) set such that every solution of the equation $\dot{x} = f(x)$ which starts in Ω remains in Ω for all future time, i.e. Ω is a *positively invariant set*. Let $V : \Omega \to I\!R$ be a continuously differentiable function such that $\frac{d}{dt}V(x) \le 0$ in Ω. Let E be the set of all points in Ω where $\frac{d}{dt}V(x) = 0$, and let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$. (For further exposition and proof see [4].)

To apply LaSalle's theorem to our system we first must find a positively invariant compact set Ω on which our Lyapunov controller V is defined. We have already established that by our choice of $\mathbf{F}$ we have $\frac{d}{dt}V \le 0$. We must then determine the set E where $\frac{d}{dt}V(\mathbf{r},\dot{\mathbf{r}}) = 0$ and determine its largest invariant set M. With this, LaSalle's theorem tells us that each trajectory starting from Ω tends when $t \to \infty$ to the largest invariant set M.

So that we may find the set Ω, we will utilize the metric d defined above and let $\bar{B}((\mathbf{x},\mathbf{y}),r) \subset I\!R^3 \times I\!R^3$ be the closed ball of radius r centered at $(\mathbf{x},\mathbf{y}) \in I\!R^3 \times I\!R^3$ in the metric d. We take the set $J = \{(\mathbf{x},\mathbf{y}) \in I\!R^3 \times I\!R^3 \mid \mathbf{x} \neq 0, |\mathrm{y}| < \mu\}$ and take an $l > 0$ so that $\bar{B}((\mathbf{L}_T,\mathbf{A}_T),l) \subset J$. Define $I = \{(\mathbf{x},\mathbf{y}) \in I\!R^3 \times I\!R^3 \mid \mathbf{x}\cdot\mathbf{y} = 0\}$, and let

$$\begin{aligned}\Omega &= \pi^{-1}(\bar{B}((\mathbf{L}_T,\mathbf{A}_T),l) \cap I)\\ &= \{(\mathbf{r},\dot{\mathbf{r}}) \in TI\!R^3_O \mid V(\mathbf{r},\dot{\mathbf{r}}) \le l^2\}.\end{aligned} \tag{4.6}$$

We must show that Ω is positively invariant and compact. To do this we recall from our work with the elliptic domain in Section 2.3 that since $\bar{B}((\mathbf{L}_T,\mathbf{A}_T),l) \cap I \subset D$, then $\Omega \subset \Sigma_e$. For all

trajectories $(\mathbf{r}, \dot{\mathbf{r}}) \in \Omega$, we recall that our Lyapunov function V is bounded above by l^2 and that $\frac{d}{dt}V(\mathbf{r}, \dot{\mathbf{r}}) \leq 0$. Thus all elliptic Keplerian orbits starting in Ω remain in Ω for all future time, and our set Ω is positively invariant. To prove compactness we first note that $\bar{B}((\mathbf{L}_T, \mathbf{A}_T), l)$ is a closed ball using our metric d and is thus compact. We also note that I is a closed set. (This is because the compliment $I' = \{(\mathbf{x}, \mathbf{y}) \in I\!R^3 \times I\!R^3 \mid \mathbf{x} \cdot \mathbf{y} \neq 0\}$ is open. We see this by taking any non-perpendicular pair of vectors $(\mathbf{x}, \mathbf{y}) \in I'$, and observing that we can find an $\epsilon > 0$ such that the neighborhood $V_\epsilon(\mathbf{x}, \mathbf{y}) = \{(\mathbf{a}, \mathbf{b}) \in I\!R^3 \times I\!R^3 | \; d((\mathbf{x}, \mathbf{y}), (\mathbf{a}, \mathbf{b})) < \epsilon\}$ contains only non-perpendicular vectors. Think of it as moving from $(\mathbf{x}, \mathbf{y})$ slightly in all directions so as to keep from being perpendicular.) We know that the intersection of a compact set and a closed set is compact, thus we have $\bar{B}((\mathbf{L}_T, \mathbf{A}_T), l) \cap I$ is a compact subset of D. We apply the following claim and conclude that $\Omega = \pi^{-1}(\bar{B}((\mathbf{L}_T, \mathbf{A}_T), l) \cap I)$ is a compact subset of Σ_e.

Claim - For any compact subset K of D, the set $\pi^{-1}(K)$ is a compact subset of Σ_e.
Proof - Let K be an compact subset of D. We know from our work with the elliptic domain that $\pi^{-1}(K)$ is a subset of Σ_e. Choose any sequence $\{a_k\} \subset \pi^{-1}(K)$ and observe that each a_k corresponds to a point on an elliptic Keplerian orbit. Let $\{b_k\} = \pi(a_k) \subset K$. Since K is compact, we know that $\{b_k\}$ has a convergent subsequence. Passing to the subindex, assume that $\{b_k\}$ converges to some $b \in K$. Recall that the fiber $\pi^{-1}(b)$ consists of a unique oriented Keplerian orbit (it is the set of all points on the orbit) which is homeomorphic to the unit circle, and thus $\pi^{-1}(b)$ is compact. By continuity of π, we know that the sequence $\{a_k\}$ converges to $\pi^{-1}(b)$. We now choose any metric on Σ_e and let $c_k \in \pi^{-1}(b)$ be a closest point from a_k to $\pi^{-1}(b)$ for each k using this metric. We know the $\{c_k\}$ is well defined because $\pi^{-1}(b)$ is compact and the distance function is continuous. Because $\pi^{-1}(b)$ is compact, $\{c_k\}$ has a convergent subsequence $\{c_{k_j}\}$ with a limit $c \in \pi^{-1}(b)$. We see then that $\{a_{k_j}\}$ converges to $c \in \pi^{-1}(b) \subset \pi^{-1}(K)$. Thus $\{a_k\}$ has a convergent subsequence and $\pi^{-1}(K)$ is compact. $\square$

Now that we have Ω a positively invariant compact set, we set out to determine the set E where $\frac{d}{dt}V(\mathbf{r}, \dot{\mathbf{r}}) = 0$ and to determine the largest invariant set M in E. We see from equations (4.4) and (4.5) that

$$E = \left\{ (\mathbf{r}, \dot{\mathbf{r}}) \in \Omega \;\middle|\; \frac{d}{dt}V(\mathbf{r}, \dot{\mathbf{r}}) = 0 \right\} = \{(\mathbf{r}, \dot{\mathbf{r}}) \in \Omega \mid \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}; \mathbf{L}_T, \mathbf{A}_T) = 0\}.$$

We define M = the largest invariant subset of E, i.e. if $(\mathbf{r}, \dot{\mathbf{r}}) \in M$ at time 0 then $(\mathbf{r}, \dot{\mathbf{r}}) \in M$ for all time. LaSalle's theorem tells us that each trajectory starting from Ω tends when $t \to \infty$ to the set M. Our goal now is to examine exactly what trajectories are contained in M. To do so we take an arbitrary trajectory $(\mathbf{r}, \dot{\mathbf{r}}) \in M$ and observe that there is no control force acting on it since $M \subset E$. Thus $(\mathbf{r}, \dot{\mathbf{r}})$ is an elliptic Keplerian orbit and we let $\mathbf{L}$ and $\mathbf{A}$ be its associated constant angular momentum and Laplace vectors. By definition of M, we note that $(\mathbf{r}, \dot{\mathbf{r}})$ are such that $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}; \mathbf{L}_T, \mathbf{A}_T) = -f(\mathbf{r}, \dot{\mathbf{r}})\, \mathbf{W} = 0$. Recall that $f(\mathbf{r}, \dot{\mathbf{r}}) > 0$, so we have

$$\mathbf{W} = \Delta\mathbf{L} \times \mathbf{r} + \mathbf{L} \times \Delta\mathbf{A} + (\Delta\mathbf{A} \times \dot{\mathbf{r}}) \times \mathbf{r} = 0. \tag{4.7}$$

Let $\Pi = span\{\mathbf{r} \times \mathbf{L}\}$ which is the plane (through the origin and perpendicular to $\mathbf{L}$) where the the ellipse swept out by $\mathbf{r}$ lies. We take the dot product of $\mathbf{r}$ with equation (4.7) and get

$$\mathbf{r} \cdot (\mathbf{L} \times \Delta\mathbf{A}) = \Delta\mathbf{A} \cdot (\mathbf{r} \times \mathbf{L}) = 0.$$

This tells us that $\Delta\mathbf{A}$ is perpendicular to the plane Π and thus parallel to the vector $\mathbf{L}$. So for some $c \in \mathbb{R}$, we have $\Delta\mathbf{A} = \mathbf{A} - \mathbf{A}_T = c\,\mathbf{L}$, noting that c is constant because both $\Delta\mathbf{A}$ and $\mathbf{L}$ are constant. Substituting into equation (4.7) we have

$$\Delta\mathbf{L} \times \mathbf{r} + (c\,\mathbf{L} \times \dot{\mathbf{r}}) \times \mathbf{r} = (\Delta\mathbf{L} - c\,(\dot{\mathbf{r}} \times \mathbf{L})) \times \mathbf{r} = 0.$$

Using the definition of $\mathbf{A} = \dot{\mathbf{r}} \times \mathbf{L} - \frac{\mu}{r}\mathbf{r}$ we then have

$$\left(\Delta\mathbf{L} - c\left(\mathbf{A} + \frac{\mu}{r}\mathbf{r}\right)\right) \times \mathbf{r} = (\Delta\mathbf{L} - c\,\mathbf{A}) \times \mathbf{r} = 0. \tag{4.8}$$

If the constant vector $\Delta\mathbf{L} - c\mathbf{A} \neq 0$, then equation (4.8) implies that this constant vector is parallel to the nonzero vector $\mathbf{r}$. But $\mathbf{r}$ changes direction with time as it sweeps out an ellipse in the plane Π. Since this is not possible, we must have $\Delta\mathbf{L} - c\mathbf{A} = 0$, which gives us

$$\mathbf{L}_T = \mathbf{L} - c\mathbf{A} \;\text{ and }\; \mathbf{A}_T = \mathbf{A} - c\,\mathbf{L}. \tag{4.9}$$

Because both $(\mathbf{L}_T, \mathbf{A}_T)$ and $(\mathbf{L},\mathbf{A})$ are contained in D, we see from equation (4.9) that

$$0 = \mathbf{L}_T \cdot \mathbf{A}_T = -c\,(L^2 + A^2).$$

Since $L \neq 0$, we must have $c = 0$ which tells us that $\mathbf{L} = \mathbf{L}_T$ and $\mathbf{A} = \mathbf{A}_T$. Thus our arbitrary orbit $(\mathbf{r}, \dot{\mathbf{r}}) \in M$ is the target orbit $\pi^{-1}(\mathbf{L}_T, \mathbf{A}_T)$. Hence the only trajectory lying in M is the target Keplerian orbit $\pi^{-1}(\mathbf{L}_T, \mathbf{A}_T)$.

We have thus shown using LaSalle's invariance principle that every trajectory starting from the subset $\Omega = \pi^{-1}(\bar{B}((\mathbf{L}_T, \mathbf{A}_T), l) \cap I)$ of $T\mathbb{R}^3_O$ remains in that subset and asymptotically converges to the target elliptic Keplerian orbit $\pi^{-1}(\mathbf{L}_T, \mathbf{A}_T)$ in our closed-loop system with the controller $\mathbf{F} = -f(\mathbf{r}, \dot{\mathbf{r}})\ \mathbf{W}$.

5. Making the Connection

We now have everything we need to complete our analysis. In Section 4, we introduced a simple feedback controller with aims of achieving asymptotically stable local orbit transfer. We used LaSalle's invariance principle to prove that when the initial orbit is within a neighborhood of the target orbit, the trajectory indeed converges towards the target orbit. We seek now to prove the same convergence using the geometric control techniques and Lie bracket calculations presented in Section 3. To do this we present the following theorem adapted from Jurdjevic-Quinn [11] stated here in the multiple input case. We will again utilize LaSalle's theorem in the proof.

Theorem 3 - Consider a smooth system on $\mathbb{R}^n$ of the form $\dot{x} = F_0(x) + \sum_{i=1}^{n} u_i(x)\,F_i(x),\ F_0(0) = 0$. We assume that:

- There exists $V : \mathbb{R}^n \to \mathbb{R}$, $V > 0$ on $\mathbb{R}^n \setminus \{0\}$, $V(x) \to +\infty$ when $|x| \to +\infty$ such that (a) $\frac{\partial V}{\partial x} \neq 0$ for $x \neq 0$, and (b) $L_{F_0}V = 0$,
- E(x)= Span$\{F_0(x), F_i(x), [F_0(x), F_i(x)], ..., ad^n F_0(F_i)\} = \mathbb{R}^n$ for *i=1,...,n* and $x \neq 0$.

Then the canonical feedback $\hat{u}_i(x) = -L_{F_i}V(x)$ for *i=1,...,n* stabilizes globally and asymptotically the origin.

Proof - We plug $\hat{u}_i(x)$ into the system and obtain the differential equation $\dot{x} = F_o(x) + \sum_{i=1}^{n} \hat{u}_i(x)\, F_i(x)$. We differentiate the function V along the trajectories of our system and have

$$\dot{V}(x) = L_{F_0 + \sum \hat{u}_i F_i} V(x) = L_{F_0} V(x) + \sum_{i=1}^{n} \hat{u}_i\, L_{F_i} V(x) = -\sum_{i=1}^{n} (L_{F_i} V(x))^2 \leq 0.$$

We shall again apply LaSalle's invariance principle which tells us that $x(t) \to M$ when $t \to +\infty$ where M is the largest invariant set in $\dot{V}(x) = -\sum_{i=1}^{n} (L_{F_i} V(x))^2 = 0$. We now evaluate the set M. First note that since M is invariant, if $x(0) \in M$ then $x(t) \in M$ for all t. We also note that on M, $\dot{V}(x) = -\sum_{i=1}^{n} (L_{F_i} V(x))^2 = 0$ and thus $L_{F_i} V(x) = \hat{u}_i = 0$ for $i = 1, ..., n$. So on M, $x(t)$ is the solution of the free motion system $\dot{x} = F_0(x)$. Thus for each i, we differentiate the equation $\dot{V}(x) = L_{F_i} V(x) = 0$ along the free motion trajectory and we get

$$\frac{d}{dt} L_{F_i} V(x) = L_{F_0} L_{F_i} V(x) = 0.$$

And since by assumption $L_{F_0} V = 0$, we deduce

$$L_{F_0} L_{F_i} V(x) - L_{F_i} L_{F_0} V(x) = L_{[F_0, F_i]} V(x) = 0, \quad \text{for i=1,...,n.}$$

Iterating the derivation we see that for all i,

$$L_{F_0} V(x) = L_{F_i} V(x) = L_{[F_0, F_i]} V(x) = L_{[F_0, [F_0, F_i]]} V(x) = ... = L_{ad^n F_0(F_i)} V(x) = 0.$$

Now we recall that $L_{F_i} V(x) = dV(x) \cdot F_i(x) = 0$ means $dV \perp F_i$ and we have that

$$M \subset \{x; \frac{\partial V}{\partial x} \perp E(x)\}.$$

Since $E(x) = I\!R^n$ for $x \neq 0$, and $\frac{\partial V}{\partial x} \neq 0$ except at $x = 0$, we conclude that $M = \{0\}$, and we have proven the theorem. □

Now we need only apply Theorem 3 to our control system to see that our analysis is complete. In our system *"the origin"* is the target orbit represented by $(\mathbf{L}_T, \mathbf{A}_T)$. We created in Section 4 the Lyapunov function $V : TI\!R^3_O \to I\!R$ equal to the square of the distance from the pair $(\mathbf{L}(\mathbf{r}, \dot{\mathbf{r}}), \mathbf{A}(\mathbf{r}, \dot{\mathbf{r}}))$ to the target $(\mathbf{L}_T, \mathbf{A}_T)$, and we proved that V satisfies the first condition of Theorem 3. In Section 3, we decomposed our system into the tangential-normal frame and proved that with a full control the orbit is the whole elliptic domain and every point of the orbit is accessible. This conclusion was made after discovering that the vectors $\mathbf{F}_0(x), \mathbf{F}_t(x), \mathbf{F}_n(x), \mathbf{F}_l(x), [\mathbf{F}_0, \mathbf{F}_c](x), [\mathbf{F}_0, \mathbf{F}_n](x)$ form a frame for the tangent space and the ad-condition is satisfied. Thus we see from our work with geometric control theory and Lie brackets that the second condition of Theorem 3 is satisfied. We may now apply the theorem above and conclude that using our feedback controller, trajectories starting in the compact set $\Omega = \pi^{-1}(\bar{B}((\mathbf{L}_T, \mathbf{A}_T), l) \cap I)$ (as given before) asymptotically converge to the target elliptic orbit. We arrive at the same stabilization result as before now utilizing the controllability techniques and Lie bracket calculations presented earlier. Observe that La Salle's invariance principle was again used in the proof of Theorem 3 above and is thus crucial to our making the connection between stability and controllability.

6. Conclusion

We have seen that geometric control theory provides valuable tools when examining mechanical systems, such as the transfer of a satellite between elliptic orbits. We focused here on the constant mass model of the controlled Kepler equation and studied only local orbit transfer. After understanding the free motion system, we made a geometric analysis of the controllability properties of the controlled Kepler equation. We studied the role of each controller in the tangential-normal frame and then established full controllability of our system. We used Lyapunov stability theory to create a simple feedback controller and establish local asymptotic stability of the target orbit. The Jurdjevic-Quinn feedback method with multiple inputs enabled us to apply our controllability results and Lie bracket computations to again prove stability of the target orbit. LaSalle's invariance principle was crucial in proving stability in both cases and in connecting the work done in [3] and [8]. An area of further study could include analyzing global orbit transfer. The idea here would be to use a finite number of intermediate orbits to transfer the satellite between arbitrary elliptic orbits. Numerical simulations should also be performed to gain an understanding of the transfer times involved. It is important to note that any numerical simulations must consider the general form of the controlled Kepler equation, which means taking into account the mass of the satellite and tracking its evolution with time.

Appendix A. Review of Mechanical Control Systems

This appendix will serve as a brief overview of simple mechanical control systems as they apply to this paper. We shall review the concepts of the free mechanical system, configuration manifolds, and the tangent space and illustrate these concepts with the example of the two-link planar manipulator (see [5] for further exposition). Our motivation is to demonstrate how we can assign to a given mechanical system a differentiable manifold which will represent the configurations of our system.

Before we delve into the mechanical system itself, we will first review some basic concepts of differential geometry, namely the charts, atlases, and differentiable structure which make up a differentiable manifold. First we shall let S be a set, and we define a *chart* for S as a pair $(\mathcal{U}, \phi)$ where (i) $\mathcal{U}$ is a subset of S and (ii) $\phi : \mathcal{U} \to I\!R^n$ is an injection (one-to-one map) for which $\phi(\mathcal{U})$ is an open subset of $I\!R^n$. We define now an *atlas* for S as a collection of charts, $\mathcal{A} = \{(\mathcal{U}_a, \phi_a)\}_{a\in\mathcal{A}}$, with the properties that $S = \cup_{a\in\mathcal{A}}\mathcal{U}_a$, and that whenever $\mathcal{U}_a \cap \mathcal{U}_b \neq \emptyset$, then (iii) $\phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$ and $\phi_b(\mathcal{U}_a \cap \mathcal{U}_b)$ are open subsets of $I\!R^n$, and (iv) the *overlap map* $\phi_{ab} \triangleq \phi_b \circ \phi_a^{-1} | \, \phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$ is a diffeomorphism from $\phi_a(\mathcal{U}_a \cap \mathcal{U}_b)$ to $\phi_b(\mathcal{U}_a \cap \mathcal{U}_b)$. (Recall that a diffeomorphism is a bijection (one-to-one and onto mapping) between open sets in $I\!R^n$ which is infinitely differentiable and its inverse is infinitely differentiable.) Basically, a chart parametrizes a subset of S, and the overlap condition makes sure that different parametrizations are compatible.

Given two atlases $\mathcal{A}_1 = \{(\mathcal{U}_a, \phi_a)\}_{a\in A}$ and $\mathcal{A}_2 = \{(\mathcal{U}_b, \psi_b)\}_{b\in B}$ for a set S, we say $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas. Thus we define a *differentiable structure* on S as an equivalence class of atlases given this equivalence relation. Finally we are able to define a *differentiable manifold*, M, as the pair (S,$\mathcal{D}$) where $\mathcal{D}$ is a differentiable structure on a set S. The topology generated by the domains of the admissible charts is called the *manifold topology* on the set S. If all charts take values in $I\!R^n$ for a fixed n, we say $n = \dim(\mathrm{M})$ is the *dimension* of M. We define a subset S of a manifold M as a *submanifold* if, for every point $x \in \mathsf{S}$, there is an admissible chart $(\mathcal{U}, \phi)$ for M with $x \in \mathcal{U}$, and such that (i) ϕ takes values in a product $I\!R^k \times I\!R^{n-k}$, and (ii) $\phi(\mathcal{U} \cap \mathsf{S}) = \phi(\mathcal{U}) \cap (I\!R^k \times \{0\})$. Basically a manifold is a set that locally looks like an open set in Euclidean space. The manifold topology allows us to take a complicated object and, with careful choice of coordinates, perform calculations on the object in Euclidean space. The structure of the manifold allows us to focus on the properties of the object as opposed to its representation in local charts.

Given this basic framework we begin to analyze the components of our mechanical system. Every mechanical system is made up of a number of *particles* (objects with mass and position but no volume) and/or *rigid bodies* (collections of particles whose relative position to one another is fixed). In order to describe the position of a particle or rigid body we must first define a space reference frame from which we measure distances and angels. We define the *spatial frame*, $\sum_{spatial} = (O_{spatial}, \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$, so that points in space can be written as vectors in $I\!R^3$ measured from the origin $O_{spatial}$ with components taken relative to the basis $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$. To specify the position of a particle we need only define a vector $\mathbf{r}$ from $O_{spatial}$ to the position of the particle. To specify the position of a rigid body on the other hand we must choose a point on the body and then specify its potion as well as its orientation relative to our spatial reference frame. Thus we must choose a *body frame* $\sum_{body} = (O_{body}, \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\})$ which moves with our rigid body. (see Figure 4) We now specify the position of our rigid body by the position vector $\mathbf{r} = O_{body} - O_{spatial}$ with components relative to our basis $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and the matrix $\mathbf{R} \in O(3) = \{\mathbf{R} \in I\!R^{3\times 3} \mid \mathbf{R}\mathbf{R}^T = \mathbf{I}_3\}$ where the i'th column contains the components of $\mathbf{b}_i$ in the basis $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$. Note that when the

bases $\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\}$ and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ share the same orientation (say right-handed), then $\mathbf{R} \in SO(3) = \{\mathbf{R} \in O(3) \mid det\ \mathbf{R} = +1\}$. For our purposes we will suppose this is the case.

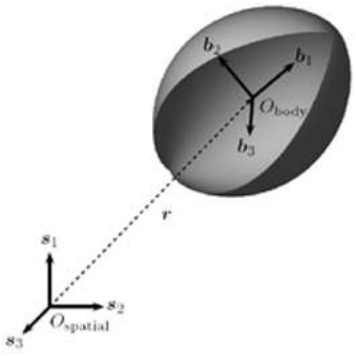


FIGURE 4. Spatial Reference Frame

We define a *free mechanical system* as a collection of N_p particles and N_b rigid bodies which move independently of one another and we write this system as $\{P_\alpha\}_{\alpha\in\{1,..,N_p\}} \cup \{B_\beta\}_{\beta\in\{1,..,N_b\}}$. Therefore the *configuration manifold for a free mechanical system* is the set

$$Q_{free} = \underbrace{(SO(3) \times I\!R^3) \times ... \times (SO(3) \times I\!R^3)}_{N_b copies} \times \underbrace{I\!R^3 \times ... \times I\!R^3}_{N_p copies}.$$

It is probably more common however that we have an interconnection between particles and rigid bodies, as is the case with our two-link manipulator. Thus an *interconnected mechanical system* is a collection of particles and rigid bodies restricted to move on a submanifold Q of Q_{free}. The manifold Q is the *configuration manifold for the interconnected system*, and if $dim(Q) = n$, the system is said to have n *degrees-of-freedom.*

We now attach to each point x in our differentiable manifold a tangent space which contains all possible "directions" in which we can pass through the point x on our manifold. The tangent space is made up of tangent vectors for each point $x \in M$. To be more precise, we first define a curve on our manifold M. For $x \in M$, a *curve at* x is a C^1-curve $\gamma : I \to M$ such that $0 \in \text{int}(I)$ and $\gamma(0) = x$. We say two curves γ_1 and γ_2 on a manifold M are *equivalent* at $x \in M$ if, for a chart $(\mathcal{U}, \phi)$ around x, the derivative of $\phi \circ \gamma_1$ and $\phi \circ \gamma_2$ are the same at 0. We define a *tangent vector* at x as an equivalence class of curves under this equivalence relation. The collection of all tangent vectors at x is the *tangent space* at x denoted T_xM, and it has the same dimension as the manifold. The *tangent bundle* is then the disjoint union of all tangent spaces and is denoted $TM = \bigcup_{x\in M} T_xM$. For our interconnected mechanical system, we had a configuration manifold Q where points in Q represented positions of the system. Now points in T_qQ represent velocities at position q, and TQ is the collection of all possible velocities at all possible positions of Q. We should note that velocities do not exist independent of positions.

To obtain a coordinate representation for velocity, we let $(\mathcal{U}, \phi)$ be a chart for our manifold M, so that points in $\mathcal{U}$ are represented by $\phi(x) \in \phi(\mathcal{U}) \subset I\!R^n$. Thus for $x \in \mathcal{U}$, we let γ be a curve which defines a tangent vector $[\gamma] \in T_xM$. Then we say the coordinate representation of that tangent

vector at $x \in \mathcal{U}$ is $(\phi(x), \mathbf{D}(\phi \circ \gamma)(0)) \in \phi(\mathcal{U}) \times I\!R^n$. We will often see the following notation:

$$\phi \circ \gamma(t) = (x^1(t), ..., x^n(t)) \text{ and } \mathbf{D}(\phi \circ \gamma)(0) = (\dot{x}^1(0), ..., \dot{x}^n(0))$$

$$\text{or alternatively } (\underbrace{(x^1, ..., x^n)}_{\text{in } \phi(\mathcal{U})}, \underbrace{(v^1, ...v^n)}_{\text{in } I\!R^n})$$

to denote the coordinate representation of a tangent vector at a point $x \in \mathcal{U}$.

Example - Two-link Planar Manipulator

We now turn to our example of the two interconnected links shown in the Figure 5. We have a fixed base point on the joint of the first link and the two links are pinned together by a second joint. We choose a spatial reference frame $(O_{spatial}, \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3\})$ with $O_{spatial}$ at the anchor point of the first link and $\mathbf{s}_3$ orthogonal to the plane of motion allowed by the joints. We choose the body reference frames of the first and second links (body 1 and body 2) as $(O_{body,a}, \{\mathbf{b}_{a,1}, \mathbf{b}_{a,2}, \mathbf{b}_{a,3}\}), a \in \{1, 2\}$ with $O_{body,a}$ at the center of mass of body a and $\mathbf{b}_{a,3}$ orthogonal to the plane of motion allowed by the joints. To simplify the system we assume that the center of mass of body 1 lies on the line between the two joints, and we choose $\mathbf{b}_{a,1}$ to point along the line connecting the joint to the center of mass of body a. Let l_1 and l_2 be the lengths of body 1 and body 2 respectively.

Because we have two rigid bodies and no particles, the configuration manifold for our free mechanical system is $Q_{free} = (SO(3) \times I\!R^3) \times (SO(3) \times I\!R^3)$, and we denote a point in Q_{free} by $((\mathbf{R}_1, \mathbf{r}_1), (\mathbf{R}_2, \mathbf{r}_2))$. We let $\theta_a \in I\!R$ be the angle $\mathbf{b}_{a,1}$ makes with $\mathbf{s}_1$. Then for body 1 we can describe the position vector $\mathbf{r}_1 = O_{body} - O_{spatial}$ and the orientation matrix

$$\mathbf{R}_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the vector $\mathbf{r}_1$ has constant length we can write $\mathbf{r}_1 = r_1\mathbf{R}_1\mathbf{s}_1$. Thus we observe that both the position and orientation of body 1 are determined by $\mathbf{R}_1 \in SO(2)$. For body 2 we have the orientation matrix

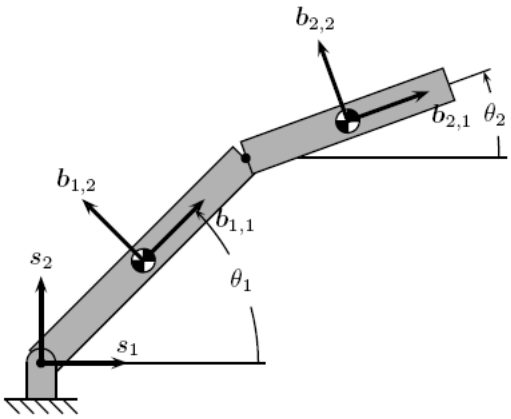
$$\mathbf{R}_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let r_2 be the distance from the second joint to $O_{body,2}$ and recall that l_1 is the distance between the two joints, thus $\mathbf{r}_2 = l_1\mathbf{R}_1\mathbf{s}_1 + r_2\mathbf{R}_2\mathbf{s}_1$. Together we have the form of a general point in $Q \subset Q_{free}$. Now observe that we can embed the special orthogonal group SO(2) × SO(2) into the submanifold $Q \subset Q_{free}$ by the map

$$\left(\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}, \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \right) \mapsto ((\mathbf{R}_1,\, r_1\mathbf{R}_1\mathbf{s}_1), (\mathbf{R}_2,\, l_1\mathbf{R}_1\mathbf{s}_1 + r_2\mathbf{R}_2\mathbf{s}_1)).$$

with $\mathbf{R}_1$ and $\mathbf{R}_2$ as defined above. Because values of θ_a differing by an integer multiple of 2π produce the same orthogonal matrix, we see that SO(2) looks like a copy of $\mathcal{S}^1$ sitting in $I\!R^{2\times 2}$. So we say the configuration manifold for our system is $Q = SO(2) \times SO(2) \cong \mathcal{S}^1 \times \mathcal{S}^1$.

FIGURE 5. Two-Link Planar Manipulator



We define a chart $(\mathcal{U}, \phi)$ for our system by

$$\begin{aligned}\mathcal{U} =& \mathcal{S}^1 \times \mathcal{S}^1 \setminus (\{((x_1, y_1), (x_2, y_2)) \in \mathcal{S}^1 \times \mathcal{S}^1 | \, x_1 = -1\} \\ & \cup \{((x_1, y_1), (x_2, y_2)) \in \mathcal{S}^1 \times \mathcal{S}^1 | \, x_2 = -1\}), \\ \phi((x_1, y_1), (x_2, y_2)) =& (\arctan(x_1, y_1), \arctan(x_2, y_2)).\end{aligned}$$

Thus we denote the coordinates for our manifold by (θ_1, θ_2) and observe that our configuration manifold is equivalent to the 2-torus. The set of points not covered by our chart is the union of two circles on the torus, as seen in Figure 6. We denote coordinates for our tangent bundle by $((\theta_1, \theta_2), (\dot{\theta_1}, \dot{\theta_2}))$.

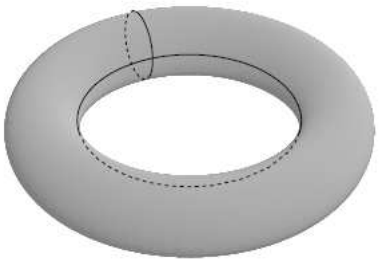


FIGURE 6. Set not covered by the chart $(\mathcal{U}, \phi)$

Appendix B. Vector Fields and the Lie Bracket

Because the Lie bracket is so fundamental to our analysis of controllability, we will spend some time here to study its properties and work through an example. In Appendix A we touched on the tangent vector, tangent space, and tangent bundle. We will begin here by looking at vector fields and flows.

A *vector field* is a smooth map that assigns a tangent vector to each point on the manifold. We may write our vector field as $X : M \to TM$ such that for all $p \in M$ we have $X(p) \in T_pM$. In coordinates, for $(x^1, ..., x^n) \in M$ we write the local representative of X as $(x^1, ..., x^n) \mapsto ((x^1, ..., x^n), (X^1, ..., X^n))$ where the $X^1, ..., X^n$ are functions, called the *components of* X, in the coordinates $(x^1, ..., x^n)$. We define an *integral curve* of a vector field X at a point $x \in M$ as a differentiable curve $\gamma : I \to M$ at x such that $\gamma'(t) = X(\gamma(t))$ for all t where γ is defined and $\gamma(0) = x$. To understand the integral curve we imagine a pebble dropped into our manifold and follow it as it moves around in the direction of the vector field. If we keep track of the time and imagine our pebble moving backward in time as well as forward, what we have is a curve $\gamma(t)$ which goes through our point at time zero, and whose derivative at each time matches the vector field. This curve is the integral curve.

It is possible that an integral curve at x cannot be defined for all t. We extend the interval on which the curve can be defined to be as large as possible and call that interval $I(X, x)$. The *maximal integral curve* of X through x is then the integral curve defined on $I(X, x)$. The *domain* of the vector field X is the set $\{(t, x) \in \mathbb{R} \times M \mid t \in I(X, x)\} = \mathrm{dom}(X)$. For $(t, x) \in \mathrm{dom}(X)$ and γ the maximal integral curve for X through x, we denote by $\Phi_t^X(x)$ the point in M given by $\gamma(t)$. We define the *flow* for X as the map $(t, x) \mapsto \Phi_t^X(x)$, and note that Φ_0^X is the identity. We imagine dropping a large number of pebbles into our manifold, one at each point in M, and letting them move in the direction of the vector field for a time t. The flow of the vector field X at time t is then the snapshot of the pebbles at time t.

We may now ask how flows compose and whether or not they commute. If we imagine our pebble moving in a vector field for time s and then again for time t, is the result the same as if the pebble were moving for time $s + t$? The answer is yes, and so long as the flow is defined for times s, t, and $s+t$, we have $\Phi_t^X \circ \Phi_s^X = \Phi_{s+t}^X$. We can write the identity $\Phi_t^X \circ \Phi_{-t}^X = \Phi_0^X = \Phi_{-t}^X \circ \Phi_t^X$ and see that the flow has a smooth inverse. Now suppose we have two vector fields, X and Y, and we drop our pebble into our manifold and have it flow with vector field X for time t and then with vector field Y for time t. Is this the same as if we first had the pebble flow with vector field Y and then X? If our vector fields were constant then the answer is yes (our flows would trace out a parallelogram). However, in general the answer is no. At each point in our manifold the pebble re-evaluates its direction according to the vector field it is following. There is no reason to believe that the vector fields will fit in a way so that the pebble will end up in the same place in the end. Thus in general vector flows are not commutative, but we can use the Lie bracket to give us some insight into the commutativity of vector flows.

To understand the Lie bracket we will first define the Lie derivative. For a vector field X and a function $f \in C^r(M)$, the *Lie derivative* of f with respect to X is the function $\mathcal{L}_X f \in C^{r-1}(M)$ defined by $x \mapsto df(x) \cdot X(x)$. In terms of our integral curve defined above, we can write the Lie derivative of f at $x \in M$ as $\mathcal{L}_X f(x) = \frac{d}{dt} f(\gamma(t))|_{t=0}$. This looks like the familiar directional

derivative of f with respect to X. In our coordinates we have

$$df = \frac{\partial f}{\partial x^i} dx^i, \quad X = X^i \frac{\partial}{\partial x^i}, \quad \text{thus} \quad \mathcal{L}_X f = X^i \, \frac{\partial f}{\partial x^i}.$$

From this representation of the Lie derivative we see that $\mathcal{L}_X f(x) = 0$ only if the function f is constant along all integral curves of M.

Given two vector fields X and Y on M, we now define the *Lie bracket* of X with respect to Y, denoted $[X, Y]$, as the unique vector field such that $\mathcal{L}_{[X,Y]} f = \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f$. In coordinates $(x^1, ..., x^n)$ we write the components of $[X, Y]^i = \sum_{j=1}^{n} (X^j \, \frac{\partial Y^i}{\partial x^j} - Y^j \, \frac{\partial X^i}{\partial x^j})$. The following are a few important properties of the Lie bracket. For all continuous functions f and g and vector fields X, Y, Z on a manifold M we have:

$$\begin{aligned}
&(i)\ [X, Y] = -[Y, X] \quad \text{(skew-symmetry)},\\
&(ii)\ [X + Y, Z] = [X, Z] + [Y, Z] \quad \text{(linearity)},\\
&(iii)\ [fX, gY] = fg[X, Y] + f(\mathcal{L}_X g)Y - g(\mathcal{L}_Y f)X \text{ , and}\\
&(iv)\ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \quad \text{(Jacobi identity)}.
\end{aligned}$$

We will sometimes use the notation $ad\, X(Y) = [X, Y]$. In fact for each integer k= 0,1,2,..., we define the vector field $ad^k X(Y)$ as follows:

$$\begin{aligned}
ad^0 X(Y) &= Y,\\
ad^1 X(Y) &= [X, Y],\\
ad^{k+1} X(Y) &= [X, ad^k X(Y)].
\end{aligned}$$

We say that a pair of vector fields (X,Y) satisfies *the ad-condition* if the linear span of all vectors of the form $ad^k X(Y)(x)$ is equal to $T_x M$, the tangent space of our manifold M at x, for each $x \in M$.

We give now perhaps one of the most important characterizations of the Lie bracket. The flow interpretation of the Lie bracket says that for vector fields X, Y of a manifold M and $x \in M$, we can define a curve γ at x by $\gamma(t) = \Phi^{-Y}_{\sqrt{t}}(x) \circ \Phi^{-X}_{\sqrt{t}}(x) \circ \Phi^{Y}_{\sqrt{t}}(x) \circ \Phi^{X}_{\sqrt{t}}(x)$. Then γ is differentiable and $\gamma\,'(0) = [X, Y](x)$. Furthermore, if X and Y are *complete*, meaning integral curves on X and Y can be extended forward and backward for all time, then $[X, Y] = 0$ if and only if $\Phi^X_t \circ \Phi^X_s = \Phi^X_s \circ \Phi^X_t$ for all times s and t. The flow interpretation tells us that if we flow along X, then Y, then -X, then -Y for the same time, and we return to the same point, then $[X, Y](x) = 0$. If $[X, Y](x) = 0$ for all points $x \in M$ then the vector fields X and Y commute. In general we shall see that $[X, Y](x) \neq 0$ and that by switching back and forth between flowing along X and Y, we can achieve a direction aligned with neither X or Y (See Figure 7).

Example - Car Parking

We will use the model of a car steering in a parking lot to demonstrate the usefulness of the Lie bracket. At a given time we can describe the position of the car by three parameters. We will use the coordinates (x, y) to describe the location of the center of mass of the car $B \in \mathbb{R}$ and the angle θ to give the orientation of the car. The driver can control the forward and backward speed of the car as well as the turning angle of the steering wheel. We can describe the motion of the car by

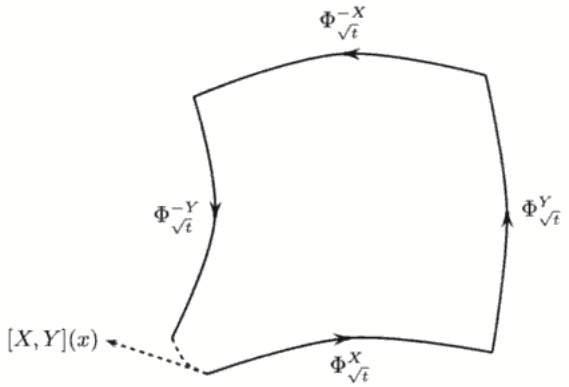


FIGURE 7. Flow interpretation of the Lie bracket

the control system

$$\begin{aligned}\dot{x} &= u_1 \cos\theta \\ \dot{y} &= u_1 \sin\theta \\ \dot{\theta} &= u_2.\end{aligned}$$

Here the velocity input u_1 corresponds to driving forward and the velocity input u_2 corresponds to steering. If we let $\mathbf{x} = (x, y, \theta)^T$ then we can write the equations of motion as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})u_1 + \mathbf{g}(\mathbf{x})u_2 \quad \text{where } \mathbf{f}(\mathbf{x}) = \begin{bmatrix}\cos x^3 \\ \sin x^3 \\ 0\end{bmatrix} \quad \text{and } \mathbf{g}(\mathbf{x}) = \begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix}.$$

We now compute the Lie bracket of the vector fields $\mathbf{f}$ and $\mathbf{g}$ as follows:

$$[\mathbf{f}, \mathbf{g}] = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}\,\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\,\mathbf{g} = \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos x^3 \\ \sin x^3 \\ 0\end{bmatrix} - \begin{bmatrix}0 & 0 & -\sin x^3 \\ 0 & 0 & \cos x^3 \\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 \\ 0 \\ 1\end{bmatrix} = \begin{bmatrix}\sin x^3 \\ -\cos x^3 \\ 0\end{bmatrix}.$$

Notice that the vector $[\mathbf{f}, \mathbf{g}](\mathbf{x})$ is orthogonal to both vectors $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$. Thus the vector field generated by the Lie bracket gives us a flow in a third direction. So by combining the flows of the vector fields $\mathbf{f}$, $\mathbf{g}$, and $[\mathbf{f},\mathbf{g}]$ we can move our system from any initial position and orientation to any final position and orientation.

We conclude this appendix with the mention of the Lie algebra. A *Lie algebra* V is an $\mathbb{R}$-vector space with the bilinear operation $[\cdot,\cdot] : V \times V \to V$ called the *bracket* which satisfies skew-symmetry (or anti-commutativity) and the Jacobi identity. In this paper we shall use the Lie bracket as the bilinear operator and will be calculating the Lie algebras generated by various vector fields in our control system.

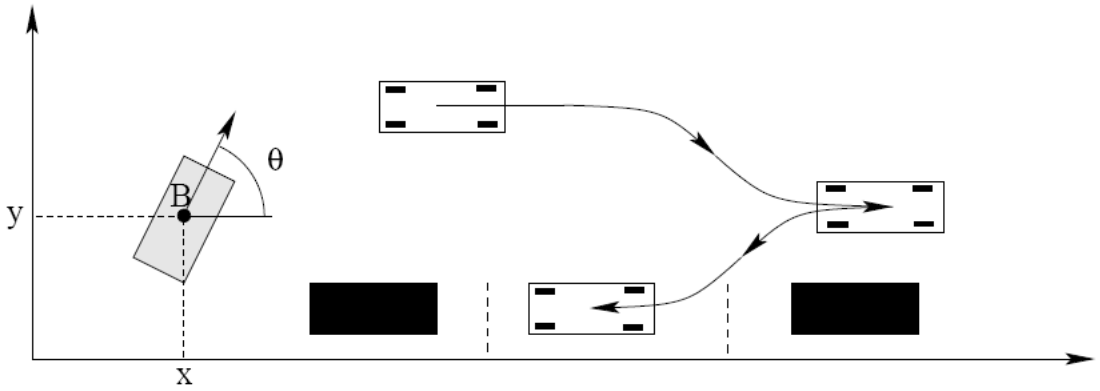


FIGURE 8. Car Parking

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