Interpolation sets, past and present – and future?

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Abstract

A survey of the theory of interpolation sets from past to present. Indications of directions for future research.

Some of this is joint work with K. E. Hare, T. W. Körner, and L. T. Ramsey.

This is a slightly emended version has of the talk originally given and has had the Beamer.cls "pauses" removed for better web display.

Outline

1. I_0 sets
   - Preliminaries
   - Hadamard sets are I_0
   - Characterizations of interpolation sets
   - I_0 subclasses–ϵ-Kronecker sets
   - I_0 subclasses–restricting the c_j's or $x(j)$s
   - Closures in $\overline{\mathbb{Z}}$
2. Appplications to Sidon sets
   - Definition and properties
   - Proportional results
3. Open questions
4. A proof

The beginning, II

Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{C}$ be bounded.

When can we extend f to be periodic on all of $\mathbb{R}$?

E compact– no problem.

Thm (Mycielski 1961, Lipinski 1960) If ε_j is a bounded sequence of real numbers and $0 \leq \lambda_j \in \mathbb{R}$ with $\lambda_{j+1}/\lambda_j \geq q > 3$ then there exists a continuous periodic function with $f(\lambda_j) = \varepsilon_j$ all j.

The use of an exponentially growing sequence was not a surprise.

The beginning, I

Hadamard gap theorem (1892). If $1 \leq n_j$, $n_{j+1}/n_j > q > 1$, then $f(z) = \sum_j c_j z^{n_j}$ is analytic precisely in a disc.

Defn:. *Hadamard (lacunary) sequence*; q is the *ratio*.

Here are 2 examples from the vast literature.

Thm. (Sidon 1927) If $\sup_x \left|\sum_{j=1}^{\infty} c_j e^{in_j x}\right| < \infty$, then $\sum_j |c_j| < \infty$.

Thm.(Banach 1930) If $\lambda_k \to \infty$, $\exists$ cts $f(x)$ such that

$$\sum_k \left|\int e^{in_k x} f(x) dx\right|^{\lambda_k} = \infty.$$

The $\int$'s are the n_k^{th} Fourier coefficients of f, and they $\to 0$.

More generally

As a species we are not good at telling if a sequence $a_n, \ n \in E$ is from the Fourier coefficients of a function, even if $E = Z$.

Therefore, work has gone into finding sets E for which, for example, **every** bounded sequence is

– extendible to a periodic (or almost periodic) function

– or the Fourier coefficients of a measure (explanations if time allows later)

Notation

$\mathbb{Z}$ – the integers $\mathbb{R}$ – the real line

$\mathbb{T}$: $[0, 2\pi]$ with addition mod 2π and 2π identified with 0

δ_x: the unit point mass at $x \in \mathbb{T}$

AP: the (Bohr) *almost periodic functions* on $\mathbb{Z}$:
the uniform limits of polynomials $p(n) = \sum_1^n c_j e^{inx(j)}$

Interpolation set (or I_0 set): $E \subset \mathbb{Z}$ such that $\ell^\infty(E) = AP_{|E}$

$\overline{\mathbb{Z}}$: The *Bohr group*: *all* group homomorphisms $\varphi : \mathbb{T} \to \mathbb{T}$, continuous or not. $\overline{\mathbb{Z}}$ is a compact group under topology of pointwise convergence, and $AP = C(\overline{\mathbb{Z}})$.

Questions and caveats

What do I_0 sets look like? Structure? Subclasses?

How "big" does E have to be to be dense in $\overline{\mathbb{Z}}$?

Everything has versions for locally compact groups in place of $\mathbb{Z}$, though complications arise if the group: has elements of finite order; is non-metrizable; or not σ-compact.

Hadamard sets are interpolation sets

Thm. (Hartman 1961) Not every bounded ϵ_j can be interpolated on 2^j with a continuous **periodic** function.

But:

Thm. (Strzelecki 1964) Every Hadamard sequence is I_0; that is, the interpolation can be done with **almost periodic** functions.

Thm. (various) If E is Hadamard, then $E \cup -E$ is I_0.

However, polynomial growth is not fast enough:

Thm. (Hartman 1961) $n_j = k^j$ for $2 \leq k$ is **not** I_0.

2 warnings

:

1. **Non-Hadamard sets:** can be I_0:

$$\{10^{k^2+j} : 1 \le j \le k < \infty\}$$

is I_0 but is not a finite union of Hadamard sets.

2. **Union difficulties:** $\{10^k :\}_{k=1}^{\infty}$ and $\{10^k + k\}_{k=1}^{\infty}$ are I_o but their union is not I_0. (Take a net $j_\alpha \to 0$ in $\overline{\mathbb{Z}}$)

The "$+k$" is essential: $\{3^k\}_{k=1}^{\infty} \cup \{3^k + 1\}_{k=1}^{\infty}$ *is* I_0.

3 early characterizations

Thm. (Mycielski 1961, Hartman and Ryll-Narzewski 1963) E is I_0 iff whenever $E = E_1 \cup E_2$ is a disjoint union, there is $\varphi \in AP$ with $\varphi = 1$ on E_1 and 0 on E_2.

Cor. (H R-N 1963) E is I_0 iff such E_1, E_2 have disjoint closures in $\overline{\mathbb{Z}}$.

Thm. (Kahane 1966) E is I_0 iff each $f \in \ell^\infty(E)$ extends to an almost periodic function $f(n) = \sum_j c_j e^{nx(j)}$, *where* $\sum_j |c_j| < \infty$.

2 more characterizations

Thm. (Kalton 1993) E is I_0 iff for each $0 < \varepsilon$ there exists $1 \leq N$ such that for each $f \in \ell^\infty(E)$ there exist $x(j) \in \mathbb{T}$ and $|c_j| \leq 1$, $1 \leq j \leq N$ such that

$$\sup_{n \in E} \left| f(n) = \sum_j c_j e^{inx(j)} \right| < \varepsilon.$$

Kalton's test is very useful, as it converts to a finite test.

Thm. (Bourgain 1984) E is I_0 iff $B_d(E) = B(E)$.

(Explanations later - if time)

ϵ-Kronecker sets, I

Def. $E \subset \mathbb{Z}$ is ε-*Kronecker* (*ε-free, C^*-embedded in $\overline{\mathbb{Z}}$*) if for every $\varphi : E \to \mathbb{T}$ there exists $x \in \mathbb{T}$ with $|\varphi(n) - e^{inx}| < \varepsilon$ on E.

Thm. (Varopoulos 1969) If $\varepsilon < 1$, and E is ϵ-Kronecker, then E is I_0.

Thm. (Hare & CG2006) If E is Hadamard with ratio $q > 3$, then E is ϵ-Kronecker for all $\varepsilon > |1 - e^{i\pi/(q-1)}|$.

Thm. (ibid.) If $\varepsilon < \sqrt{2}$, and E is ϵ-Kronecker, then E is I_0.

ϵ-Kronecker sets, II

Thm.(Ibid) If $0 \leq \varepsilon < \sqrt{2}$ and E is ϵ-Kronecker, then $E = \{n_k\}$ with $n_{k+1} - n_k \to \infty$

Thm. (Ibid.) There exists a $\sqrt{2}$-Kronecker set that is not I_0.

FZI_0 sets

Def. E is FZI_0 if every bounded Hermitian function on E is of the form $\sum_j c_j e^{inx(j)}$, where $0 \leq c_j$ and $\sum c_j < \infty$.

(Hermitian: $f(-n) = \overline{f(n)}$ whenever $n, -n \in E$.)

Thm. (GH 2006-2008) E is FZI_0 iff
a) E is I_0,
b) $0 \notin E$, and
c) $E \cup -E$ is I_0.

There exist I_0 sets for which c) fails.

Hadamard sets and ε-Kronecker sets are FZI_0 ($\varepsilon < \sqrt{2}$).

$I_0(U)$ sets, I

Let $U \subset \mathbb{T}$.

Def. E is $I_0(U)$ if every bounded function on E can be interpolated by a sum $\sum c_j e^{inx(j)}$, where $x(j) \in U$ and $\sum |c_j| < \infty$.

E is $I_0(U)$ *with bounded constants* if there exists C such that for each open $U \subset \mathbb{T}$ there exists a finite set $F \subset E$ such that every bounded function on $E \backslash F$ can be interpolated by a sum $\sum_j c_j e^{inx(j)}$, where $x(j) \in U$ and $\sum_j |c_j| \leq C$.

$x(j)$ restricted to small sets, II

Thm. (Méla 1969) Every I_0 set is a finite union of $I_0(U)$ sets with bounded constants.

Thm. (GH 2005) Hadamard sets are $I_0(U)$ with bounded constants

Thm. (GH 2005) ϵ-Kronecker sets are $I_0(U)$ with bounded constants ($\varepsilon < \sqrt{2}$)

Thm. (GH 2005) If E is $I_0(U)$ with bounded constants, then $E + F$ is I_0 for all finite sets F. (There are I_0 sets E with $E \cup (E + 1)$ not I_0.)

Thm. (GHR 200?) Every I_0 set is $I_0(U)$ for every open U.

Closures

Thm. (Ryll-Narzewski 1964, Ramsey 1980) If E is I_0, then E does not accumulate at elements of $\mathbb{Z}$.

Thm. (GHK 2006) If $0 \leq \varepsilon < 2$ and E is ϵ-Kronecker, then the closure of E in $\overline{\mathbb{Z}}$ has no interior points. (E is not always I_0)

Thm.(GHK 2006) (i). If $\varepsilon < \sqrt{2}$, and E is ϵ-Kronecker, then the only Bohr cluster point of $E - E$ in $\mathbb{Z}$ is 0.

(ii). If $\varepsilon < 2\sin(\pi/(8M))$, then $\overbrace{E + \cdots + E}^{M}$ has no Bohr cluster points in $\mathbb{Z}$.

Definition of Sidon sets

Recall Sidon's theorem of 1927:
If $\sup_x \left|\sum_{j=1}^{\infty} c_j e^{in_j x}\right| < \infty$, then $\sum_j |c_j| < \infty$.

Sidon's Thm gave rise to a defn:
E is *Sidon* if for all continuous $f = \sum_{n\in E} c_n e^{inx}$, we have $\sum_{n\in E} |c_n| < \infty$.

Properties

Thm. TFAE:
(i). E is Sidon;
(ii). For every $\varphi : E \to \mathbb{C}$ with $\varphi \to 0$ there exists $f \in L^1(\mathbb{T})$ with $\varphi(n) = \int e^{-int} f(t)dt,$ for $n \in E$.
(iii). For every bounded $\varphi : E \to \mathbb{C}$ there is a bounded Borel measure μ with $\varphi(n) = \int e^{-int} d\mu(t),$ for $n \in E$; and

Cor. I_0 sets are Sidon.

There are Banach space reformulations of (i)-(iii).

Union Thm; quasi-independent sets, I

Thm. (Drury 1970) The union of 2 Sidon sets is Sidon.

Def. E is *quasi-independent* if whenever $1 \le N$, $e_j = 0, \pm 1$, $n_j \in E$, and $\sum_1^N e_j n_j = 0$ we have $e_j = 0$, $1 \le j \le N$.

Thm.(Classical) Quasi-independent sets are Sidon.

Pisier's Thm

Def. *E* is *proportionally* quasi-independent if there exist $0 < \delta$ such that for every finite $F \subset E$, there is a quasi-independent set H such that $H \subset F$ and $\#H \geq \delta \# F$.

Thm. (Pisier 1982) *E* is Sidon iff it is proportionally quasi-independent.

Cor. The union of 2 Sidon sets is Sidon.

Kalton's Thm

Recall Kalton's Thm:

Thm. E is I_0 iff for each $0 < \varepsilon$ there exists $1 \leq N$ such that for each $f \in \ell^\infty(E)$ there exist $x(j) \in \mathbb{T}$ and $|c_j| \leq 1$, $1 \leq j \leq N$ such that

$$\sup_n \left| f(n) = \sum_{j=1}^{N} c_j e^{inx(j)} \right| < \varepsilon.$$

We say that E is $I_0(N, \varepsilon)$

Thm. (Ramsey 1996) E is Sidon iff it is proportionally $I_0(N, \varepsilon)$ for some $1 \leq N$ and $0 < \varepsilon < 1$.

Other proportionality results

Thm. (GH 2008) TFAE.
(i) E is Sidon.
(ii). E is proportionally ϵ-Kronecker.
(iii). E is proportionally $FZI_0(N, \varepsilon)$ for some $1 \leq N$ and $0 < \varepsilon < 1$.
(iv.) E is cofinitely proportional $FZI_0(N, \varepsilon, U)$ for all open U.

(I will save "cofinitely" for questioners.)

Questions

1. If E is ϵ-Kronecker and $\sqrt{2} \leq \varepsilon < 2$, is E Sidon? (It's not necessarily I_0 – G-H 2006)
2. Is every quasi-independent set I_0?
3. Is every Sidon set a finite union of I_0 sets?
4. Can a Sidon set be dense in $\overline{\mathbb{Z}}$?
   (Yes to #3 implies No to #4.)

Kahane's Theorem and sketch of a proof, I

Thm. (Kahane 1966) E is I_0 iff each $f \in \ell^\infty(E)$ extends to an almost periodic function $f(n) = \sum_j c_j e^{nx(j)}$, *where* $\sum_j |c_j| < \infty$.

Proof. Let $\Omega = \{z : |z| = 1\}^E$, the set of functions on E whose values are in $\{z : |z| = 1\}$. With the product topology, Ω is a compact metric space.

With coordinate-wise multiplication, Ω is an abelian group. Of course, the multiplication is continuous.

For $1 \leq N$ and $0 < \varepsilon$, let $\Omega(N, \varepsilon)$ be the set of $\omega \in \Omega$ for which there exist $x(1), \ldots, x(N) \in \mathbb{T}$ and $c_1, \ldots, c_N \in \mathbb{C}$ with $|c_j| \leq 1$ and $|\omega(n) - \sum_j c_j e^{inx(j)}| \leq \varepsilon$ for all $n \in E$.

Sketch of proof, II

Then
a) $\Omega(N, \varepsilon)$ is closed in Ω,

b) $\Omega(N, \varepsilon) \subset \Omega(N + 1, \varepsilon)$ for $1 \leq N$, and

c) $\Omega(N, \varepsilon) \cdot \Omega(M, \delta) \subset \Omega(N + M, \varepsilon + \delta + \varepsilon\delta)$. (Exercise.)

Sketch of proof, III

Since every bounded function on E extends to an AP function, and since the AP functions are uniform limits of trig polys (finite sums $\sum_1^N c_j e^{inx(j)}$),
d) $\bigcup_N \Omega(N, \varepsilon) = \Omega$ for all $0 < \varepsilon$.

The Baire category theorem says one of the $\Omega(N, \varepsilon)$ has non empty interior.

Since Ω is a compact group, a finite number of translates of $\Omega(N, \varepsilon)$ cover Ω.

Sketch of proof, IV

Let $\omega_k + \Omega(N, \varepsilon)$, $1 \leq k \leq K$ be such translates. For each k there exists $N(k)$ such that $\omega_k \in \Omega(N(k), \varepsilon)$.

Let $M = \max N(k)$. Then $\Omega \subset \Omega(M + N, 2\varepsilon + \varepsilon^2)$ by c).

$\frac{1}{2}\Omega + \frac{1}{2}\Omega$ contains all functions on E bounded by 1, so we have shown

There exists L such that for every f bounded by 1 on E, there exists c_j and $x(j)$ such that

$$|f(n) - \sum_1^L c_j e^{inx(j)}| \leq \varepsilon \text{ for all } n \in E \text{ and } \sum_j c_j| \leq L.$$

That proves Kalton's theorem.

Sketch of proof, V

A standard iteration shows that every bounded $f : E \to \mathbb{C}$ is of the form

$$f(n) = \sum_{1}^{\infty} c_j e^{inx(j)}, \text{ where } \sum_j |c_j| < \infty.$$

And that proves Kahane's theorem.