Summation methods that are (and are not) support-perceiving for pseudofunctions and pseudomeasures

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Abstract

A general theorem and a general principle unify classical approaches to localization (support-perception) for trigonometric series and integrals.

Known results become clearer and have (in some cases) easier proofs. New results are suggested.

This approach covers functions, pseudofunctions, measures, and pseudomeasures, but does not extend to all distributions on T^d and R^d, leading to some open questions.

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Outline

1. A theorem and a heuristic principle
   - General context and definitions
   - That theorem
   - That principle
2. New results
   - Theorem 1 and pseudofunctions
   - The heuristic principle and pseudomeasures
3. Localization for distributions
   - Beyond pseudomeasures
   - Open issues
4. Sketch of proofs

Context

G: a metrizable nondiscrete LCA group (e.g. T^d or R^d)
Γ: dual group (e.g., Z^d or R^d)
$\langle x, \gamma \rangle$ for $x \in G$, $\gamma \in \Gamma$: the pairing.
$A(G)$: $\{\widehat{F} : F \in L^1(\Gamma)\}$
$PM(G)$: **pseudomeasures**: $\{S : \widehat{S} \in L^\infty(\Gamma)\} = A(G)^*$.
W: Banach space with $W \subset PM(G)$ such that

$\widehat{f} \cdot W \subset W$ for all $\widehat{f} \in A(G)$ (can define $Supp\, w$)

$\widehat{f} * W \subset W$ for all $f \in A(G)$ (can ask: TP dense in W?)

Define "$f * S$" as $(f\widehat{S})^\vee$ if needed.

Examples:
$L^1(G)$, $C(G)$ for compact G, and
$PF(G)$–the **pseudofunctions**: $\{S : \widehat{S} \in C_0(\Gamma)\}$.

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Summation method: sequence $L_N \in L^1(\Gamma) \cap L^\infty(\Gamma)$ with $L_N \to 1$ pointwise on compacta and $\sup_N \|L_N\|_\infty < \infty$.

(Don't need $0 \leq L_N$ here.)

Pointwise localization holds for $\{L_N\}$ and W if for all $S \in W$ and $x \notin \mathit{Supp}\, S$

$$\widehat{L_N} * S(x) = \int_\Gamma \widehat{S}(\gamma) L_N(\gamma) \langle x, \gamma \rangle d\gamma \to 0.$$

Uniform localization: unif in compacta disjoint from $\mathit{Supp}\, S$

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Let $f(x) \sim \sum_n c_n e^{inx} \in L^1(T)$

Behaviour of $\sum_n c_n e^{inx}$ at x_0 depends only on $f(x)$ in arbitrarily small nbhds of x_0.

Equivalent to: $\sum_n c_n e^{inx} \to 0$ off of $Supp\, f$.

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is bounded for all nbhds U of $0 \in G$.

Corollary 2. *Localization occurs for L_N and $W = L^1(G)$ iff for all nbhds U of $0 \in G$*

$$\sup\{|L_N(x)| : x \notin U, 1 \leq N\} < \infty.$$

Particular forms of Cor. 2 appear in Bochner (1936, p.194) and in Mayer (1967, p.678) for $SU(2)$, but the general version has not been exploited systematically.

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Corollary 2 implies...

For $S \in L^1$:

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  local. holds because
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A Cor. to Thm 1:

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(i). Yields RRZ loc. for T.S. and $PF(T)$.

(ii). Uniform localization equivalent to pointwise.

(iii). Covers all standard summation methods, + others.

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Theorem 5. *Let $\{L_N\}$ be a summation method on Γ. Then pointwise localization holds for all $S \in PM(G)$ iff for all open neighbourhoods U of $0 \in G$,*

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(i) Uniform localization equivalent to pointwise.
(ii). Includes (C,1) means for $PM(XT)$ and $PM(R)$.
(iii). Algebra gives loc. for $L_N(t) = \chi_{[-N,N]}(1 - |t|^k/N^k)$ on $PM(R)$.

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Known distribution results

- If $\widehat{S}(t) = O(|t|^k)$ for $0 \leq k < 1$ then localization holds for (C,1) means ($G = R$). (GV 2000)
- If $\widehat{S}(t) = O(|t|^k)$ for $0 \leq k < \infty$ then localization holds for (C,k+1) means (k integer, $G = R$). (CCG 2007)
- If $\widehat{S}(t) = O(|t|^k)$ for $0 \leq k < \infty$ then localization holds for Bochner-Riesz means of order $\alpha = \alpha(k, d)$ (k integer, $G = R^d$). (GV-CCG 2007)
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- When does pointwise loc. imply uniform loc.?

  δ_0' is instructive: all (C,k) means $\to 0$ pointwise everywhere!!
- Find general statements for distributions S with $\widehat{S}(t) = o(|t|^k)$? (or $O(|t|^k)$).
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Restatement of Thm 1 preparatory to proof

To prove:
(A). For all $S \in W$,

$$\lim_{N\to\infty} \int \widehat{S}(\gamma) L_N(\gamma)\langle x, \gamma\rangle d\gamma = 0 \text{ for all } x \notin \mathit{Supp}\, S.$$

$$\Longleftrightarrow$$

(B). For all $\varphi \in L^1(\Gamma)$ with $\widehat{\varphi} = 1$ in a nbhd of $0 \in G$,

$$\sup_N \|\widehat{L_N} - \widehat{\varphi}\widehat{L_N}\|_{A(G)} < \infty.$$

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(B) $\Rightarrow$ (A)

Assume (B). By translation wolog $x = 0$. Choose φ so $Supp\,\widehat{\varphi} \cap Supp\,S = \emptyset$. Then $\widehat{\varphi}S = 0$, so we must show that

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It is not hard to show that

Lemma. $L_N - \varphi * L_N \to 0$ uniformly on compacta.

A $2 - \varepsilon$ argument using the Lemma, $\widehat{S} \in C_0(\Gamma)$, and $\|L_N - \varphi * L_N\|_1 \leq C$ gives (A).

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