Finitely Presented Expansions of Algebras

(Algebraic Specifications of Abstract Data Types)

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Definition: A many sorted algebra is a tuple $\boldsymbol{A}=(A_1, A_2,\ldots A_n; f_1,f_2,\ldots, f_k, c_1,\ldots,c_r)$, where

each A_i is a set of elements of a given sort,
each f_j is a function of type $(s_1,\ldots,s_m)\rightarrow s$.
each c_k is a constant.

Definition: Terms are defined as follows. Variables of each sort and constants are terms.

Suppose that $t_1,\ldots,t_m$ are terms, f is a function of type $(s_1,\ldots,s_m)\rightarrow s$, where the range of each t_i is of sort s_i.

Then the expression $f(t_1,\ldots,t_m)$ is a term.

The set of ground terms can be turned into a many sorted algebra called

the term algebra

denoted by F. The algebra is finitely generated and computable.

Fact: Every many sorted algebra generated by the constants is a homomorphic image of F.

An ***equational specification*** is a finite set of formulas of type

$$t=q;$$

A ***quasiequational specification*** is a finite set of formulas of type

$$t_1=q_1 \& \ldots\ldots \& t_n=q_n \rightarrow t=q,$$

where t,q are terms that may contain variables.

Definition: (Specified Algebras)

Let E(S) be the congruence relation generated by S.

The algebra F_S = F/ E(S) is called specified by S.

F_S is equationally specified if S is an equational specification.

F_S is quasiequationally specified if S is a quasiequational specification.

Properties of F_S:

1. F_S satisfies S and is finitely generated.
2. The equality relation E(S) of F_S is computable enumerable.
3. The algebra F_S is universal.
4. The algebra F_S is unique.
5. If S is equational then all homomorphic images of F_S satisfy S.

Specification Problem:

Can a given many sorted algebra ***A*** be specified?

We need to at least assume that ***A*** is f.g. and the equality relation in ***A*** is c.e.

We single out these into the following definition.

Definition

A ***computably enumerable algebra*** is one of the form F/E, where

F is the free term algebra,
E is a computably enumerable congruence relation on F.

Example: specified algebras.

A start up example:

The algebra (ω, S, f), where $f(x)=2^x$, is not be specified in its own language.

An ***important*** observation is, however, this:

Consider the expansion (ω, S, f, +, x). This expanded algebra is now specified.

Definition: (Computable Algebra)

A many sorted algebra

$$\boldsymbol{A}=(A_1, A_2, \ldots A_n; f_1, f_2, \ldots, f_k, c_1, \ldots, c_r)$$

is ***computable*** if all sets A_i and functions f_j are computable.

Definition: An ***expansion*** of

$$\boldsymbol{A}=(A_1, A_2,\ldots A_n; f_1,f_2,\ldots., f_k, c_1,\ldots,c_r)$$

is

$$(A_1, A_2,\ldots A_n; f_1,f_2,\ldots., f_k, g_1,\ldots,g_r, c_1,\ldots,c_r),$$

where $g_1,\ldots,g_r$ are new functions.

Theorem (Bergstra & Tucker, ≈1980).

Every computable many sorted algebra has an equationally specified expansion.

Definition: An ***extension*** of

$$\boldsymbol{A}=(A_1,\ldots A_n;\ f_1,\ldots.,\ f_k,\ c_1,\ldots,c_r)$$

is a many sorted algebra:

$$(A_1,\ldots A_n,C_1,..,C_m;\ f_1,\ldots.,\ f_k,g_1,..,g_h,\ c_1,\ldots,c_r)$$

where $C_1,..,C_m$ are new sort sets, and $g_1,..,g_h$ are new functions.

Theorem (Bergstra & Tucker, ≈1980)

Any many sorted c.e. algebra has an equationally specified extension.

This theorem positively solves the specification problem if one allows to ***extend*** many sorted c.e. algebras.

Problem

(early 1980s, Bergstra & Tucker, Goncharov)

Let ***A*** be a finitely generated computably enumerable algebra:

1. Does ***A*** have an equationally specified expansion?
2. Does ***A*** have a quasiequationally specified expansion?

Resemblance to Higman’s theorem

Theorem

(Kassymov, 1988; Khoussainov, 1994)

There exists a finitely generated computably enumerable algebra no expansion of which can be equationally specified.

Essential facts used in the proof of this result are properties 3, 4 and 6 of specified algebras.

Many attempts have been done to fully solve the problem, especially in the late 80s and early 90s. See surveys on algebraic specifications during those years.

The main theorem:

(Goncharov, Hirschfeldt, Khoussainov)

There exists a finitely generated computably enumerable algebra no expansion of which can be quasiequationally specified.

Proof involves several concepts and results that are interesting on their own. We explain them below.

Fix the language (f,g,h,…; a,b,c,..) with finitely many function and constant symbols.

Let F be the algebra of ground terms.

The desired algebra ***A*** we construct will be a homomorphic image of F. The algebra will be constructed by stages.

At stage n, some ground terms t_n and g_n are glued. Thus, at this stage we have a finite set of ground term equations

$$E_n = \{ t_1=g_1, t_2=g_2, \ldots, t_n=g_n \}.$$

The algebra $\boldsymbol{A}_n$ is obtained by factorizing F by E_n. The desired algebra will be the limit of this sequence of algebras.

Definition: An ***almost free*** algebra is the factor of the free algebra F by a finite set of ground term equations.

To construct the desired algebra satisfying the main theorem we need to have a good understanding of almost free algebras.

Theorem (Kozen, 1976)

1) The word problem for almost free algebras is decidable.

2) It is also decidable whether a given almost free algebra is finite.

Implicitly in Kozen (1976):

An algebra is almost free if and only if it is the free term extension of a finite partial algebra.

Free extensions: Let B be a partial algebra.
The free extension of B is denoted by Free(B).

The free extension Free(B) is a total algebra.

Example:

The free extension of (Constants; f,g,h,…), where the functions are defined nowhere, is the free algebra F.

Theorem (Khoussainov, Rubin)

*Let **B** be a decidable partial algebra. Then its free extension Free (**B**) is also a decidable algebra.*

Corollary. *Every almost free algebra is decidable.*

This extends Kozen's result to FO logic.

Now, we can control the sequence of almost free algebras

$$\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2, \ldots.$$

in order to build the desired algebra $\boldsymbol{A}$.

Here is an example. Say, we want $\boldsymbol{A}$ not to be specified in its own language.

CONSTRUCTION 1:

We list all the specifications

$$S_0, S_1, S_2, \ldots\ldots$$

We want ***A*** not to be isomorphic to F_S for every specification S.

Thus, we have infinitely many requirements.

Set $\boldsymbol{A}_0$=F. Take S_0. Check if $\boldsymbol{A}_0$ satisfies S_0.

If $\boldsymbol{A}_0$ satisfies S_0 then take two terms t_0,q_0, glue them, and $\boldsymbol{A}_1$ is the result obtained by factorizing $\boldsymbol{A}_0$ w.r.t. t_0=q_0.

Otherwise, take witnesses that do not satisfy S_0, and ensure that these witnesses never glued. Set $\boldsymbol{A}_1$=$\boldsymbol{A}_0$.

Take S_1. Check if $\boldsymbol{A}_1$ satisfies S_1.

If $\boldsymbol{A}_1$ satisfies S_1 then take two terms t_1,q_1, glue them, and $\boldsymbol{A}_2$ is the result obtained by factorizing $\boldsymbol{A}_1$ w.r.t. $t_1=q_1$.

Otherwise, take witnesses that do not satisfy S_1, and ensure that these witnesses never glued. Set $\boldsymbol{A}_2=\boldsymbol{A}_1$. Continue this on inductively.

Let

$$\boldsymbol{A}=(F/E;\ f,g,\ldots)$$

be a f.g. computably enumerable algebra.

Take any computable function

$$h: F^n \to F.$$

If the action of h on F/E is well-defined then consider the expansion:

$$(F/E;\ f,g,\ldots,\ h)$$

Definition: The function h is a ***term-like*** if there is a term t of the original algebra ***A*** such that

$$h(b)=t(b)$$

for all b in $(F\setminus X)^n$, where X is a finite set.

Definition: (Primal Algebra)

Let

$$\boldsymbol{A}=(F/E;\ f,g,\ldots)$$

be an a f.g. computably enumerable infinite algebra. We say that ***A*** is ***primal*** if all expansions of ***A*** by computable functions are term-like.

Theorem

(Goncharov, Hirschfeldt, Khoussainov)

There exists a finitely generated computably enumerable infinite primal algebra.

Proof is based on a finite injury priority argument construction. Call it

CONSTRUCTION 2

Corollary. *Assume that* **A** *is a finitely generated c.e. algebra such that*

a) ***A*** *is not specified in its own language.*
b) ***A*** *is primal.*

*Then no expansion of **A** is specified.*

Theorem

(Goncharov, Hirschfeldt, Khoussainov)

*There exists a finitely generated c.e. algebra **A** such that:*

*a) **A** is not specified in its own language.*
*b) **A** is primal.*

The proof is based on putting the two constructions together.

Conclusion:

We have a full picture of results related to the specification problem:

1. Computable algebras can be specified if expansions are allowed.
2. Computably enumerable algebras can be specified if extensions are allowed.
3. Expansions are not powerful enough to specify computably enumerable algebras.