Multiplication Operators, Riemann Surfaces and Analytic continuation

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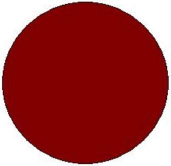
This is a joint work with
Ronald G. Douglas and Shunhua Sun.

Bergman space

- Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$:



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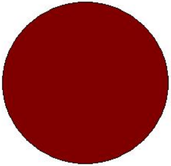

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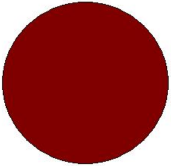

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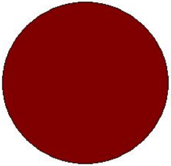

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- The Bergman space

$$L^2_a = \{f \in L^2(\mathbb{D}, dA) : f \quad \text{is analytic in } \mathbb{D}\}$$

Reproducing kernel

- Inner product

$$\begin{aligned} \langle f, g \rangle &= \int_{\mathbb{D}} f(z)\overline{g(z)}\,dA(z) \\ &= \sum_{n=0}^{\infty} \frac{a_n \overline{b_n}}{n+1} \end{aligned}$$

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- $$k_z(w) = \sum_{j=0}^{\infty} e_n(w)\overline{e_n(z)} = \frac{1}{(1-\overline{z}w)^2}.$$

Multiplication Operators

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- M_z is the Bergman shift
$$M_z e_n = \sqrt{\frac{n+1}{n+2}} e_{n+1}.$$

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- $$\sigma(M_\phi) = \overline{\phi(\mathbb{D})}.$$

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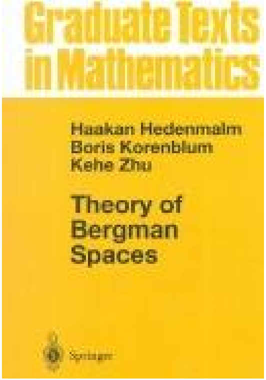

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On the Hardy Space

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Every invariant subspace $\mathcal{M}$ of the multiplication operator T_z by z is equal to θH^2 for some inner function θ.

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On the Bergman Space

Aleman-Richter-Sundberg Theorem

Every invariant subspace $\mathcal{M}$ of the multiplication operator M_z by z is generated by its wandering subspace $\mathcal{M} \ominus M_z\mathcal{M}$., i.e.,

$$\mathcal{M} = [\mathcal{M} \ominus M_z\mathcal{M}].$$

Index Theorem

For each n equal to positive integer or $+\infty$, there is an invariant subspace $\mathcal{M}$ of M_z such that

$$dim\ [\mathcal{M} \ominus M_z\mathcal{M}] = n.$$

Multiplication operators?

On the Hardy Space

Lattice T_z:

$\{\theta H^2 : \text{inner } \theta\}$.

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Lattice M_z?

$\Downarrow$ **Apostol-Bercovici-Foias-Pearcy**

Invariant subspace problem

Does every bounded operator on a Hilbert space have an invariant subspace?

Multiplication Operators via Complex Geometry

Cowen-Douglas classes

If ϕ is a rational function with poles outside of the closed unit disk,

$$M_\phi^* \in \mathcal{B}_n(\Omega)$$

for some n and open set $\Omega \subset \mathbb{C}$.

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$T \in \mathcal{B}_n(\Omega)$ on a Hilbert space $\mathcal{H}$ if

- $\Omega \subset \sigma(T)$.
- $Ran(T - \omega) = \mathcal{H}$ for each $\omega \in \Omega$.
- $\bigvee_{\omega \in \Omega} ker(T - \omega) = \mathcal{H}$.
- $dimker(T - \omega) = n$ for each $n \in \Omega$.

Multiplication Operators via Theory of Subnormal Operators

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Suppose S_1 and S_2 are two subnormal operators on a Hilbert space H and N_1 and N_2 are the minimal normal extensions of S_1 and S_2 on K respectively. If S_1 and S_2 are unitarily equivalent, i.e., $W^*S_1W = S_2$ for some unitary operator W on H, then there is a unitary operator $\tilde{W}$ on K such that

$$W = \tilde{W}|_H, \qquad \tilde{W}^*N_1\tilde{W} = N_2.$$

Minimal Normal Extension of M_ϕ

For ϕ in H^∞, let $\tilde{M}_\phi$ denote the multiplication operator by ϕ on $L^2(\mathbb{D}, dA)$ given by $\tilde{M}_\phi g = \phi g$ for each g in $L^2(\mathbb{D}, dA)$.

- *The minimal normal extension of the Bergman shift M_z is the operator $\tilde{M}_z$ on $L^2(\mathbb{D}, dA)$.*
- *For each finite Blaschke product ϕ, the minimal normal extension of M_ϕ is the operator $\tilde{M}_\phi$ on $L^2(\mathbb{D}, dA)$.*

Thus for each unitary operator W commuting with M_ϕ, there is a unitary operator $\tilde{W}$ on $L^2(\mathbb{D}, dA)$ such that

$$W = \tilde{W}|_{L^2_a}, \qquad \tilde{W}^* \tilde{M}_\phi \tilde{W} = \tilde{M}_\phi.$$

Multiplication Operators via $H^2(\mathbb{D}^2)$

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Then $\mathcal{H}$ is the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^{\infty}$.
- Define the operator $U : L_a^2(D) \to \mathcal{H}$ by
$$Uz^n = \frac{p_n(z,w)}{n+1}.$$
U is a unitary operator from L_a^2 onto $\mathcal{H}$.

Lifiting the Bergman shift

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- Thus the Bergman shift is lifted up as the compression of an isometry on a nice subspace of $H^2(\mathbb{T}^2)$.
- For any inner function ϕ,

$$M_\phi \cong \phi(\mathcal{B}) = P_{\mathcal{H}} T_{\phi(z)}|_{\mathcal{H}} = P_{\mathcal{H}} T_{\phi(w)}|_{\mathcal{H}}.$$

Problem 1

Classify reducing subspaces of M_ϕ?

Finite Blaschke product:

$$\phi(z) = \prod_{j=1}^{n} \frac{z - \alpha_i}{1 - \bar{\alpha}_i z},$$

where α_i are points in the unit disk $\mathbb{D}$.

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• A reducing subspace $\mathcal{M}$ for an operator T on a Hilbert space H is a subspace $\mathcal{M}$ of H such that $T\mathcal{M} \subset \mathcal{M}$ and $T^*\mathcal{M} \subset \mathcal{M}$.

Commutant

For a bounded operator S, the commutant

$$\{S\}' = \{T \in B(L_a^2) : ST = TS\}$$

Let

$$\mathcal{A}_\phi = \{M_\phi\}' \cap \{M_\phi^*\}'.$$

Then

$$\mathcal{A}_\phi = \{T : TM_\phi = M_\phi T, TM_\phi^* = M_\phi^* T\}.$$

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So $\mathcal{A}_\phi$ is a von Neumann algebra.

If $\mathcal{M}$ is a reducing subspace of M_ϕ and $P_\mathcal{M}$ is the orthogonal projection from the Bergman space onto $\mathcal{M}$, then

$$P_\mathcal{M} M_\phi = M_\phi P_\mathcal{M}, \quad P_\mathcal{M} M_\phi^* = M_\phi^* P_\mathcal{M}.$$

Problem 2

Structure of the von Neumann algebra $\mathcal{A}_\phi$?

Why finite Blaschke product ϕ?

Theorem (Cowen-Thomson)

For each rational function f with poles outside of the closed unit disk, there is a finite Blaschke product ϕ such that

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• The local inverses $\{\rho_j\}_{j=1}^n$ of an n-th order Blaschke product ϕ have group-like property under composition.

• The Riemann surface $\phi^{-1} \circ \phi$ over $\mathbb{D}$ is completely determined by the polynomial $P(w)Q(z) - P(z)Q(w)$ where $P(z)$ an $Q(z)$ are coprime polynomials of two complex variables z and w such that

$$\phi(z) = \frac{P(z)}{Q(z)}.$$

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Theorem

There exists a unique reducing subspace $\mathcal{M}_0(\phi)$ for $\phi(\mathcal{B})$ such that $\phi(\mathcal{B})|_{\mathcal{M}_0(\phi)}$ is unitarily equivalent to the Bergman shift. In fact,

$$\mathcal{M}_0(\phi) = \bigvee_{l\geq 0} \{p_l(\phi(z), \phi(w))e_0\} \subset \mathcal{H},$$

and $\{\frac{p_l(\phi(z),\phi(w))e_0}{\sqrt{l+1}\|e_0\|}\}_0^\infty$ *form an orthonormal basis of* $\mathcal{M}_0(\phi)$*. Here* $e_0(z, w) = \frac{\phi(z)-\phi(w)}{z-w}$.

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Theorem

For a function ϕ in H^∞, if M_ϕ on the Bergman space has the distinguished reducing subspace on which the restriction of M_ϕ is unitarily equivalent to the Bergman shift, then ϕ must be a finite Blaschke product.

Distinguished Reducing Subspace $\mathcal{M}_0(\phi)$

Theorem (Stessin-Zhu)

If ϕ is a finite Blaschke product and $\phi(0) = 0$,

$$\mathcal{M}_0(\phi) \approx \bigvee_{n=0}^{\infty} \phi^n \phi'.$$

Second Order Blaschke products

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Theorem (Sun-Wang-Zhu)

Let $\phi(z) = \prod_{j=1}^{2} \frac{z-\alpha_i}{1-\bar{\alpha}_i z}$. Then M_ϕ has only two nontrivial minimal reducing subspaces $\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}$.

Third Order Blaschke products via $H^2(\mathbb{D}^2)$

Theorem 1 (Guo-Sun-Zheng-Zhong)

Let ϕ be a Blaschke product with three zeros.
• If $\phi(z)$ has a multiple critical point in the unit disk, (i.e., $\phi \approx z^3$) then the lattice of reducing subspaces of $M_\phi \approx$ the lattice generated by

$$\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\},$$

where $\mathcal{M}_j = \bigvee_{k=0}^{\infty}\{z^{3k+j}\}$ for $j = 1, 2, 3$.

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• If ϕ does not have any multiple critical point in the unit disk, then the lattice of reducing subspaces of M_ϕ is generated by $\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}$.

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• If ϕ does not have any multiple critical point in the unit disk, then the lattice of reducing subspaces of M_ϕ is generated by $\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^{\perp}\}$.

By **Bochner Theorem** that every Blaschke product with n zeros has exactly $n-1$ critical points in the unit disk $\mathbb{D}$, we obtain the classification of reducing subspaces of M_ϕ for a Blaschke product ϕ with the third order.

Fourth order Blaschke products

Theorem (Sun-Zheng-Zhong)

Let ϕ be a fourth order Blaschke product. (1) If $\phi \approx z^4$, the lattice of reducing subspaces of $M_\phi \approx$ the lattice generated by

$$\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4\},$$

where $\mathcal{M}_j = \bigvee_{k=0}^{\infty}\{z^{4k+j}\}$ for $j = 1, 2, 3, 4$.
(2) If $\phi \not\cong z^4$ but ϕ is decomposable, i.e., $\phi = \psi_1 \circ \psi_2$ for two Blaschke products ψ_1 and ψ_2 with orders 2 then the lattice of reducing subspaces of M_ϕ is generated by

$$\{\mathcal{M}_0(\phi), \mathcal{M}_0(\psi_2) \ominus \mathcal{M}_0(\phi), \mathcal{M}_0(\psi_2)^\perp\}.$$

(3) If ϕ is not decomposable, then the lattice of reducing subspaces of M_ϕ is generated by

$$\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}.$$

Reducing Subspaces vs "Geometry'

Theorem 1 (Guo-Sun-Zheng-Zhong)

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• If $\phi(z)$ has a multiple critical point in the unit disk, (i.e., $\phi \approx z^3$) then the lattice of reducing subspaces of $M_\phi \approx$ the lattice generated by

$$\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\},$$

where $\mathcal{M}_j = \bigvee_{k=0}^{\infty}\{z^{3k+j}\}$ for $j = 1, 2, 3$.

• If ϕ does not have any multiple critical point in the unit disk, then the lattice of reducing subspaces of M_ϕ is generated by $\{\mathcal{M}_0(\phi), \mathcal{M}_0(\phi)^\perp\}$.

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Reducing Subspaces vs "Geometry'

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Problem 3

Can we classify reducing subspaces by 'geometry' of ϕ?

Conjecture

For a Blaschke product ϕ of finite order, the number of nontrivial minimal reducing subspaces of M_ϕ equals the number of connected components of the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$.

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Let ϕ be a finite Blaschke product. The von Neumann algebra $\mathcal{A}_\phi$ is generated by $\mathcal{E}_1, \cdots, \mathcal{E}_q$ and has dimension q. Here q is the number of connected components of the Riemann surface $\phi^{-1} \circ \phi$ over the unit disk and

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- The Riemann surface $\phi^{-1} \circ \phi$ over the unit disk.
- G_{i_k} is the collection of local inverses of ϕ which are mutually analytically continuable.
- Local inverses
- Analytical continuation

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Let ϕ be a finite Blaschke product with order less than or equal to 8. Then $\mathcal{A}_\phi$ is commutative and hence, in these cases, the number of minimal reducing subspaces of M_ϕ equals the number of connected components of the Riemann surface $\phi^{-1} \circ \phi$ over the unit disk.

ABELIAN C^*-ALGEBRAS

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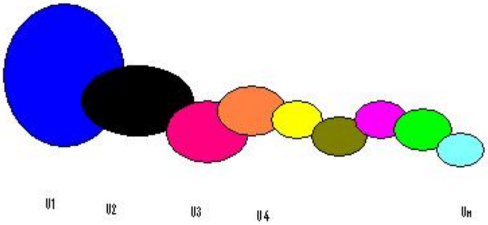
Order $= 2, 3, 4.$

Guo-Huang: *Order* $= 5, 6.$

Analytic continuation

An analytic function element is a pair (f, U), which consists of an open disk U and an analytic function f defined on this disk.
A finite sequence $\mathcal{U} = \{(f_j, U_j)\}_{j=1}^m$ is a continuation sequence if

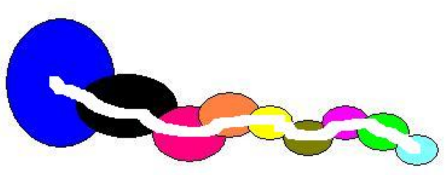
- $U_j \cap U_{j+1}$ is not empty for $j = 1, \cdots, m-1$ and
- $f_j \equiv f_{j+1}$ on $U_j \cap U_{j+1}$, for $j = 1, \cdots, m-1$.



Analytic continuation

Let γ be an arc with parametrization $\gamma(t)$, $\gamma(t)$ being a continuous function on an interval $[0,1]$. A sequence $\{U_1, \cdots, U_m\}$ is admissible or a covering chain for γ if each U_j is an open disk, and if there exist increasing numbers $t_1, \cdots t_m$ in $[0,1]$ such that $\gamma(t_j) \in U_j$ for $j = 1, \cdots, m$ and

$$\gamma(t) \in \begin{cases} U_1, & 0 \leq t \leq t_1 \\ U_j \cup U_{j+1}, & t_j \leq t \leq t_{j+1} \\ U_m, & t_m \leq t \leq 1. \end{cases}$$



A continuation sequence $\mathcal{U} = \{(f_j, U_j)\}_{j=1}^{m}$ is an analytic continuation along the arc γ if the sequence $U_1, \cdots, U_m$ is admissible for γ. Each of $\{(f_j, U_j)\}_{j=1}^{m}$ is an analytic continuation of the other along the curve γ. We say that the analytic function f_1 on U_1 admits a continuation in U_m.

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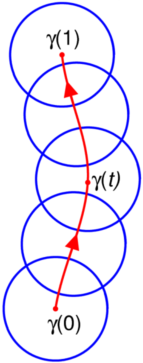

Theorem (Riemann Monodromy Theorem)

Suppose $\Omega \subset \mathbb{C}$ is a simply connected open set. If an analytic element, (f, U) can be analytically continued along any path inside Ω, then this analytic function element can be extended to be a single-valued holomorphic function defined on the whole of Ω.

Local Inverses

Bochner's theorem says that ϕ has exactly $n - 1$ critical points in the unit disk $\mathbb{D}$ and none on the unit circle.

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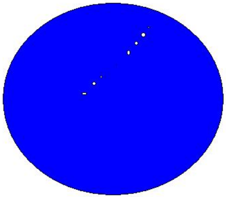
- Let

$$E = \mathbb{D}/\mathcal{F}.$$

For an open set $V \subset \mathbb{D}$, we define a **local inverse** of ϕ in V to be a function f analytic in V with $f(V) \subset \mathbb{D}$ such that $\phi(f(z)) = \phi(z)$ for every z in V. That is, f is a branch of $\phi^{-1} \circ \phi$ defined in V.



• Not all of the branches of $\phi^{-1} \circ \phi$ can be continued to a different branch, for example z is a single valued branch of $\phi^{-1} \circ \phi$.
• A local inverse (f, V) admits an analytic continuation along the curve γ in E if there is a continuation sequence $\mathcal{U} = \{(f_j, U_j)\}_{j=1}^{m}$ admissible for γ and (f_1, U_1) equals (f, V).
• A local inverse in $V \subset E$ is admissible for ϕ if it admits unrestricted continuation in E.
• As $\phi : E \to \mathbb{D}$ is an n-to-one mapping , for each $z \in E$, $\phi^{-1} \circ \phi(z)$ contains n distinct points $\{z_1, \cdots, z_n\}$ in E.

• For each $z \in E$, $\phi(\rho(z)) = \phi(z)$ has n solutions $\phi^{-1} \circ \phi = \{\rho_k(z)\}_{k=1}^{n}$ on an open neighborhood D_z of z

• (ρ_j, D_z) is an analytic element and it is locally analytic and arbitrarily continuable in E.

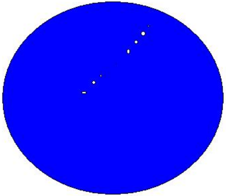
• $\{\rho_j\}_{j=1}^{n}$ is the family of admissible local inverses in some invertible open disc $V \subset \mathbb{D}$.

For a point $z_0 \in V$, label those local inverses as $\{\rho_j(z)\}_{i=1}^n$ on V.

If there is a loop γ in E at z_0 such that ρ_j and $\rho_{j'}$ in $\{\rho_i(z)\}_{i=1}^n$ are mutually analytically continuable along γ, we can then write

$$\rho_j \sim \rho_{j'},$$

and it is easy to check that $\sim$ is an equivalence relation.



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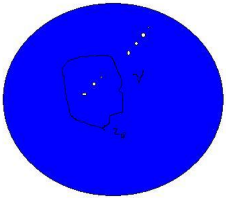

Using the equivalence relation, we divide $\{\rho_i(z)\}_{i=1}^n$ into equivalence classes

$$\{G_{i_1}, G_{i_2}, \cdots, G_{i_q}\}$$

where $i_1 = 1 < i_2 < i_3 < \cdots < i_q \le n$. Each element in G_{i_k} extends analytically to the other in G_{i_k}, but can not extend to any element in G_{i_l} if $i_k \ne i_l$.

$\{\rho_j\}_{j=1}^{n}$ FORM A GROUP UNDER COMPOSITION.

The set of admissible local inverses has the useful property that is closed under composition

$$\rho \circ \hat{\rho}(z) = \rho(\hat{\rho}(z)).$$

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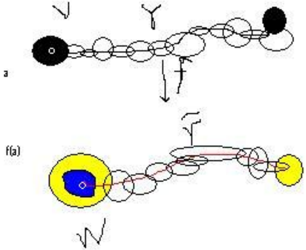
Is $\rho \circ \hat{\rho}$ arbitrarily continuable in E?

Let f and g be admissible local inverses in open discs V and W centered at a and $f(a)$, respectively, with $f(V) \subset W$.

Let γ be a curve in E with initial point a. f can be analytically continued along γ. There is an obvious image curve $\tilde{\gamma}$ of γ under this analytic continuation along γ. g can be analytically continued along $\tilde{\gamma}$. By refining the covering chain of γ, we can assume that if $\tilde{V}$ is a covering disc of γ and $(\tilde{f}, \tilde{V})$ the corresponding function element, then $\tilde{f}(\tilde{V})$ is contained in one of the covering discs of $\tilde{\gamma}$. We now compose corresponding function elements in the analytic continuations along γ and $\tilde{\gamma}$ to obtain an analytic continuation for $(g \circ f, V)$ along γ.



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Every element in a von Neumann algebra $\mathcal{A}$ can be written as a finite linear combination of unitary operators in $\mathcal{A}$.

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Representation of unitary operators

For each unitary operator W in the von Neumann algebra $\mathcal{A}_\phi$, we hope to get a nice representation of W on the Bergman reproducing kernel k_α.

Local Representation

Theorem

Let ϕ be a finite Blaschke product. Let U be an invertible open set of E. Then for each T in $\{M_\phi\}'$, there are analytic functions $\{s_i(\alpha)\}_{i=1}^n$ on U such that for each h in the Bergman space L_a^2,

$$\begin{aligned} Th(\alpha) &= \sum_{i=1}^n s_i(\alpha) h(\rho_i(\alpha)), \\ T^* k_\alpha &= \sum_{i=1}^n \overline{s_i(\alpha)} k_{\rho_i(\alpha)} \end{aligned}$$

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This follows easily from that *ker* $M^*_{\phi-\phi(\alpha)} = \bigvee_{j=1}^n k_{\rho_j(\alpha)}$.

Riemann surfaces of $\phi^{-1} \circ \phi$ over $\mathbb{D}$

A finite Blaschke product ϕ with n zeros is an n to 1 conformal map of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$.

• Let $\phi = \frac{P(z)}{Q(z)}$ be a Blaschke product of order n where $P(z)$ and $Q(z)$ are two coprime polynomials of degree n.

• Let $f(w, z) = P(w)Q(z) - P(z)Q(w)$.

Then $f(w, z)$ is a polynomial of w with degree n and of z. For each $z \in \mathbb{D}$, $f(w, z) = 0$ has exactly n solutions in $\mathbb{D}$.

The Riemann surface S_ϕ equals the locus S_f of solutions of the equation $f(w, z) = 0$ in $\mathbb{D}^2$ except for a few branch points.
• The Riemann surface S_ϕ for $\phi^{-1} \circ \phi$ over $\mathbb{D}$ is an n-sheeted cover of $\mathbb{D}$ with at most $n(n-1)$ branch points, and it is **not connected.**

By unique factorization theorem for the ring $\mathbb{C}[z,w]$ of polynomials of z and w, we can factor $f(w,z) = \prod_{j=1}^{q} p_j(w,z)^{n_j}$ where $p_1(w,z), \cdots, p_q(w,z)$ are irreducible polynomials. Bochner's Theorem says that ϕ has finite critical points in the unit disk $\mathbb{D}$. Thus we have $f(w,z) = \prod_{j=1}^{q} p_j(w,z)$.

Theorem

Let $\phi(z)$ be an n-th order Blaschke product and $f(w,z) = \prod_{j=1}^{q} p_j(w,z)$. Suppose that $p(w,z)$ is one of factors of $f(w,z)$. Then the Riemann surface S_p is connected if and only if $p(w,z)$ is irreducible. Hence q equals the number of connected components of the Riemann surface $S_\phi = S_f$.

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Remark

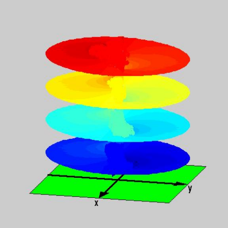
q equals the number of irreducible components of the plane algebraic curve determined by

$$f(w,z) = 0.$$

Visualizing Riemann Surfaces

Visualization of Riemann surfaces is complicated by the fact that they are embedded in C^2, a four-dimensional real space. One aid to constructing and visualizing them is a method known as **cut and paste.**

We begin with n copies of the unit disk $\mathbb{D}$, called sheets. The sheets are labeled $\mathbb{D}_1, \cdots, \mathbb{D}_n$ and stacked up over $\mathbb{D}$.



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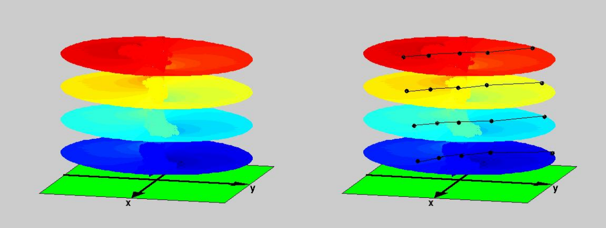
Let $\{z_1, \cdots, z_m\}$ be the branch points. Suppose a curve Γ drawn through those branch points and a fixed point on the unit circle so that $\mathbb{D}/\Gamma$ is a simply connected region.



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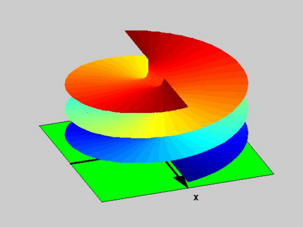
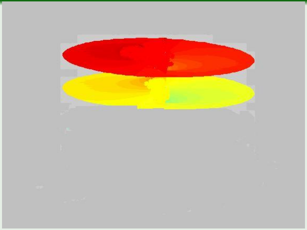
Various sheets are glued to others along opposite edges of cuts. With the point in the k-th sheet over a value z in $\mathbb{D}/\Gamma$ we associate the pair of values $(\rho_k(z), z)$. In this way a one-to-one correspondence is set up between the points in S_f over $\mathbb{D}/\Gamma$ and the pair of points on the n sheets over $\mathbb{D}/\Gamma$.



Example

If $\phi(z) = \prod_{j=1}^{2} \frac{z-\alpha_i}{1-\bar{\alpha}_i z}$, the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$ has two connected components..



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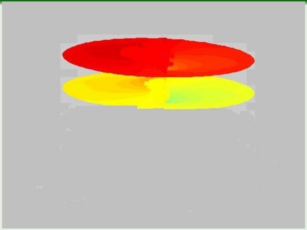

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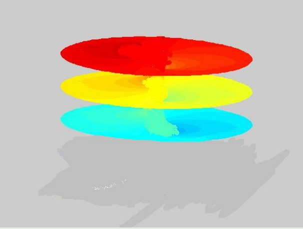
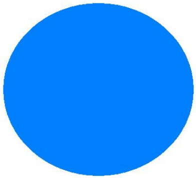
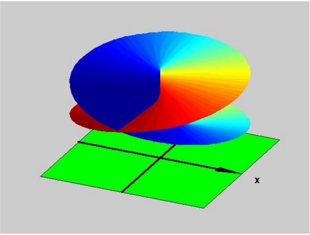
If $\phi(z) = z^3$, then the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$ has three connected components.



Example (Stephenson)

If $\phi(z) = z^2 \frac{z-\alpha}{1-\bar{\alpha}z}$ for $\alpha \neq 0$, the Riemann surface of $\phi^{-1} \circ \phi$ over $\mathbb{D}$ has only two connected components.



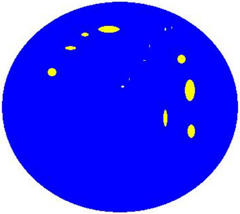


We need to order $\{\rho_j\}_{j=1}^n$ globally over a simply connected subset of E. To do this, take an invertible small open set U of E such that the intersection of $\rho_j(U)$ and $\rho_k(U)$ is empty for $j \neq k$. We always can do so by shrinking U sufficiently. Fix the small invertible open set U and label $\{\rho_j(z)\}_{j=1}^n$ as

$$\{\rho_1(z), \rho_2(z), \cdots, \rho_n(z)\}$$

and assume $\rho_1(z) = z$.

Take a curve Γ through the finite set $\mathcal{F}$ and connecting a point on the unit circle so that $E\backslash\Gamma$ is simply connected and disjoint from the set $\cup_{j=1}^n \rho_j(U)$.



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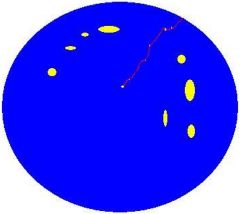

LABEL $\{\{\rho_j(z)\}_{j=1}^n, U\}$

The Riemann Monodromy theorem gives that each of $\{\rho_j(z)\}_{j=1}^n$ has a uniquely analytic continuation on $E\backslash\Gamma$ and hence we can view each of $\{\rho_j(z)\}_{j=1}^n$ as an analytic function on $E\backslash\Gamma$ satisfying

$$\phi(\rho_j(z)) = \phi(z),$$

We still order $\{\rho_j(z)\}_{j=1}^n$ as $\{\rho_1(z), \rho_2(z), \cdots, \rho_n(z)\}$ at for $z \in E\backslash\Gamma$ and $j = 1, \cdots, n$. every point in $E\backslash\Gamma$. The label is denoted by $\{\{\rho_j(z)\}_{j=1}^n, U\}$.

Global representation

Theorem

Let ϕ be a finite Blaschke product. Let U be a small invertible open set of E. Let $\{\rho_i(z)\}_{j=1}^n$ a complete collection of local inverses and $E\backslash\Gamma$ with the label $\{\{\rho_j(z)\}_{j=1}^n, U\}$. Then for each T in $\{M_\phi\}'$, there are analytic functions $\{s_i(\alpha)\}_{i=1}^n$ on $E\backslash\Gamma$ such that for each h in the Bergman space L_a^2,

$$\begin{aligned} Th(\alpha) &= \sum_{i=1}^n s_i(\alpha)h(\rho_i(\alpha)), \\ T^* k_\alpha &= \sum_{i=1}^n \overline{s_i(\alpha)} k_{\rho_i(\alpha)} \end{aligned}$$

for each α in $E\backslash\Gamma$.

Representation of unitary operators

If W is a unitary operator in $\{M_\phi\}'$, then W^* is also in $\{M_\phi\}'$.

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$$\begin{aligned} Wh(\alpha) &= \sum_{i=1}^n r_i \rho_i'(\alpha) h(\rho_i(\alpha)), \\ W^* k_\alpha &= \sum_{i=1}^n \overline{r_i}\, \overline{\rho_i'(\alpha)} k_{\rho_i(\alpha)} \end{aligned}$$

for each α in $E\backslash\Gamma$.

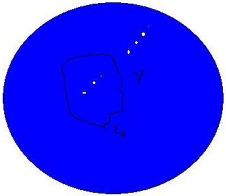
Theorem

Let U be a small invertible open set of E. Let $\{\rho_i(z)\}_{j=1}^{n}$ be a complete collection of local inverses. Then for each α in U,

$$W^* k_\alpha = \sum_{k=1}^{q} \hat{r}_k \sum_{\rho \in G_{i_k}} \overline{\rho'(\alpha)} k_{\rho(\alpha)},$$

where $\hat{r}_k = r_\rho$ for some ρ in G_{i_k}.

Take a curve γ through a point z_0 in E. The analytic continuation along γ of $\overline{W^* k_\alpha}$ is still $W^* k_\alpha$, but for each $\rho \in G_{i_k}$, the analytic continuation of $\overline{\rho'(\alpha) k_{\rho(\alpha)}}$ will turn to $\overline{\hat{\rho}'(\alpha) k_{\hat{\rho}(\alpha)}}$ for some $\hat{\rho} \in G_{i_k}$



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where $\hat{r}_k = r_\rho$ for some ρ in G_{i_k}.

For each $1 \leq k \leq q$, define a bounded linear operator $\mathcal{E}_k : L_a^2 \to L_a^2$ by

$$\mathcal{E}_k f(z) = \sum_{\rho \in G_{i_k}} \rho'(z) f(\rho(z))$$

for $z \in E$ and each $f \in L_a^2$.

$$W^* = \sum_{k=1}^{q} \hat{r}_k \mathcal{E}_k^*.$$

Proof of Main Result

Russo-Dye Theorem

Every element in a von Neumann algebra $\mathcal{A}$ can be written as a finite linear combination of unitary operators in $\mathcal{A}$.

For each unitary operator $W \in \mathcal{A}_\phi$,

$$W = \sum_{k=1}^{q} \overline{\hat{r}_k} \mathcal{E}_k.$$

The von Neumann algebra $\mathcal{A}_\phi$ is generated by $\mathcal{E}_1, \cdots, \mathcal{E}_q$ and its dimension $\leq q$ if each $\mathcal{E}_k$ is in $\mathcal{A}_\phi$.

The dimension of $\mathcal{A}_\phi$ equals q if $\mathcal{E}_1, \cdots, \mathcal{E}_q$ are linearly independent.

Problems

$$\mathcal{E}_k f(z) = \sum_{\rho \in G_{i_k}} \rho'(z) f(\rho(z)).$$

- Is $\mathcal{E}_k$ well-defined?

- $\mathcal{E}_k \in \mathcal{A}_\phi$?

What is $\mathcal{E}_k^*$?

- Are $\mathcal{E}_1, \cdots, \mathcal{E}_q$ linearly independent?

$\mathcal{E}_k$ IS WELL-DEFINED AND COMMUTES WITH M_ϕ

For each polynomial f, $\sum_{\rho\in G_{i_k}} \rho'(z)f(\rho(z))$ extends analytically on E and for each ρ.

$$\begin{aligned}\mathcal{E}_k(M_\phi f) &= \sum_{\rho\in G_{i_k}} \rho'(z)\phi(\rho(z))f(\rho(z)) \\ &= \phi(z)\sum_{\rho\in G_{i_k}} \rho'(z)f(\rho(z)) \\ &= M_\phi\mathcal{E}_k(f).\end{aligned}$$

Thus

$$\mathcal{E}_k M_\phi = M_\phi \mathcal{E}_k.$$

Change Variable Formula

To get $\mathcal{E}_k^*$, we need the following change variable formula.

Lemma

For each $\rho \in \{\rho_j\}_{j=1}^n$ *and* $f \in L_a^2$,

$$\int_{E\backslash\Gamma} |f(\rho(z))|^2 |\rho'(z)|^2 dA(z) \quad = \quad \int_{\mathbb{D}} |f(w)|^2 dA(w).$$

This follows from that for each $\rho \in \{\rho_j\}_{j=1}^n$, $\rho(E\backslash\Gamma)$ contains $\mathbb{D}\backslash\hat{\Gamma}$ where $\hat{\Gamma}$ denotes the set

$$\Gamma \cup \cup_{j=1}^n \{w \in E\backslash\Gamma : \rho_j(w) \in \Gamma\},$$

which consists of finitely many curves on $\mathbb{D}$. Since ρ maps E into E and is locally analytic and injective on E, $\hat{\Gamma}$ is a closed set which area measure equals zero.

$\mathcal{E}_k^* = \mathcal{E}_{k^-}$

Since ρ^{-1} is also in $\{\rho_i\}_{i=1}^n$ for each $\rho \in \{\rho_i\}_{i=1}^n$, let $G_{i_k}^-$ denote the subset of $\{\rho_i\}_{i=1}^n$:

$$\{\rho : \rho^{-1} \in G_{i_k}\}.$$

We need the following lemma to get $\mathcal{E}_k^*$.

Lemma

For each i_k, there is an integer k^- with $1 \leq k^- \leq q$ such that

$$G_{i_k}^- = G_{i_{k^-}}.$$

Theorem

For each integer k with $1 \leq k \leq q$, there is an integer k^- with $1 \leq k^- \leq q$ such that

$$\mathcal{E}_k^* = \mathcal{E}_{k^-}.$$

$\mathcal{E}_1, \cdots, \mathcal{E}_q$ ARE LINEARLY INDEPENDENT.

Assume that there are constants $c_1, \cdots, c_q$ such that

$$c_1\mathcal{E}_1 + \cdots + c_q\mathcal{E}_q = 0.$$

Thus for each α in E, we have

$$[c_1\mathcal{E}_1 + \cdots + c_q\mathcal{E}_q]^* k_\alpha = 0.$$

For each i, define

$$P_i(\alpha, z) = \prod_{j \neq i}^{n} (z - \rho_j(\alpha)).$$

An easy calculation gives

$$\langle P_{i_k}(\alpha, .), [c_1\mathcal{E}_1 + \cdots + c_q\mathcal{E}_q]^* k_\alpha \rangle = c_k \rho'_{i_k}(\alpha) P_{i_k}(\alpha, \rho_{i_k}(\alpha)).$$

Since $P_{i_k}(\alpha, \rho_{i_k}(\alpha)) \neq 0$ and $\rho'_{i_k}(z)$ vanishes only on a countable subset of $\mathbb{D}$, we have that c_k must be zero for each k. We conclude that $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_q$ are linearly independent

THEOREM

Let ϕ be a finite Blaschke product with order less than or equal to 8. Then $\mathcal{A}_\phi$ is commutative.

Let ϕ be a finite Blaschke product. Let G be those local inverses of ϕ which extend analytically to only themselves in $\mathbb{D}\backslash\mathcal{F}$.

LEMMA

G is an elementary subgroup of Aut($\mathbb{D}$) consisting of elliptic Möbius transforms and identity ρ_1. Moreover, there are a point α in $\mathbb{D}$, a unimodular constant λ and an integer n_G such that

$$G = \{\frac{|\alpha|^2 - \lambda}{1 - |\alpha|^2\lambda}\phi_{\frac{\alpha(1-\bar{\lambda})}{1-|\alpha|^2\bar{\lambda}}}(z) : \lambda^{n_G} = 1\}.$$