Abstract

For every countable group G, a family of isomorphism invariants for measure-preserving G-actions on probability spaces is defined. In the special case in which G is a countable sofic group, a special class of these invariants are computed exactly for Bernoulli systems over G. This leads to a complete classification of Bernoulli systems for many countable groups including all finitely generated linear groups. These results are combined with recent rigidity results of S. Popa to obtain classification results for Bernoulli shifts over special classes of groups G up to von Neumann equivalence and/or orbit equivalence.

Isomorphism invariants for actions of sofic groups

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1 Introduction

Let G be a countable group. A G**-system** is a probability space (X, μ) together with a measure-preserving action of G on this space. Two such systems (X, μ) and (Y, ν) are **isomorphic** if there are conull sets $X' \subset X, Y' \subset Y$ and a measurable map $\phi : X' \to Y'$ with measurable inverse $\phi^{-1} : Y' \to X'$ such that $\phi(gx) = g\phi(x) \forall g \in G, x \in X'$. A function f from the set of G-systems to say, the real numbers, is an **isomorphism invariant** if the value of $f(X, \mu)$ depends only on the isomorphism class of (X, μ).

This paper introduces, for each countable group G, a family of isomorphism invariants. However, it appears that these invariants are useful only in the special case in which G is a countable sofic group. Sofic groups were introduced by Gromov in [Gr99] (although he called them 'initially subamenable'; the term 'sofic' was coined in [We00]). The class of sofic groups contains all residually amenable groups and therefore, all linear groups. It is not known whether or not every countable group is sofic. For more results on sofic groups, see [ES06]. A precise definition is given in section 7.

The new invariants are used to classify a large class of Bernoulli systems, which are defined as follows. Let (K, κ) be a probability space. Let K^G denote the set of all functions $x : K \to G$. Alternatively, K^G can be considered as the product space $\prod_{g \in G} K$ of G copies of K. Let κ^G be the product measure on K^G. G acts on K^G by $(g_1 x)(g_2) = x(g_1^{-1} g_2)$ for all $g_1, g_2 \in G$ and $x \in K^G$. This action preserves the measure κ^G. The system (K^G, κ^G) is called the **Bernoulli system over G with base** (K, κ).

The **entropy** of the base measure κ is defined as follows. If there exists a finite or

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countable subset $K' \subset K$ with $\kappa(K') = 1$ then

$$H(\kappa) := -\sum_{k \in K'} \kappa(\{k\}) \log(\kappa(\{k\})).$$

Otherwise $H(\kappa) = +\infty$. Here we are using the convention $0 \log(0) = 0$ and the base of the logarithm is e.

The main application of the results in this paper is the following theorem.

Theorem 1.1. *Let G be a countable sofic group and let $(K_1, \kappa_1), (K_2, \kappa_2)$ be two probability spaces such that both K_1 and K_2 are finite. If (K_1^G, κ_1^G) is isomorphic to (K_2^G, κ_2^G) then $H(\kappa_1) = H(\kappa_2)$.*

We will say that a group G is an **Ornstein group** if the converse to the above theorem holds but without the finiteness restriction. That is, G is **Ornstein** if whenever $(K_1, \kappa_1), (K_2, \kappa_2)$ are two probability spaces with $H(\kappa_1) = H(\kappa_2)$ then (K_1^G, κ_1^G) is isomorphic to (K_2^G, κ_2^G). Ornstein proved in [Or70a, Or70b] that $\mathbb{Z}$ is an Ornstein group. This result was extended to a large class of amenable groups in [OW87] including every countably infinite amenable group. Stepin proved in [St75] that if G contains a subgroup H which is Ornstein then G itself is Ornstein. For example, any group that contain an infinite cyclic subgroup is Ornstein. A complete proof of Stepin's result is given in [Bo08]. The above theorem implies the following corollary in the case in which K_1 and K_2 are finite.

Corollary 1.2. *Let G be a countable sofic Ornstein group. Let $(K_1, \kappa_1), (K_2, \kappa_2)$ be probability spaces. Then (K_1^G, κ_1^G) is isomorphic to (K_2^G, κ_2^G) if and only if $H(\kappa_1) = H(\kappa_2)$.*

By contrast, in many cases, any two nontrivial Bernoulli shifts are weakly isomorphic. Here are the definitions. A factor map between G-systems $(X_1, \mu_1), (X_2, \mu_2)$ is a measurable map $\phi : X_1' \to X_2$ (where $X_1' \subset X_1$ is conull) such that $\phi(gx) = g\phi(x) \forall g \in G, x \in X_1'$ and $\phi_* \mu_1 = \mu_2$. In this case, we say that (X_2, μ_2) is a **factor** of (X_1, μ_1). The systems (X_1, μ_1) and (X_2, μ_2) are **weakly isomorphic** if each is a factor of the other.

Theorem 1.3. *Let G be a countable group that contains a nonabelian free group. Let $(K_1, \kappa_1), (K_2, \kappa_2)$ be any two probability spaces with $H(\kappa_1)H(\kappa_2) > 0$. Then (K_1^G, κ_1^G) is weakly isomorphic to (K_2^G, κ_2^G).*

This theorem follows immediately from the case in which G is a free group (proven in [Bo08]) and the theory of coinduced actions (see the second to last section of [Bo08]). In section 9, this is used to prove:

Theorem 1.4. *Let G be a countable group that contains a nonabelian free group. Let (K, κ) be a probability space with $H(\kappa) = +\infty$. Then there are no finite generating partitions for (K^G, κ^G).*

Of course, the conclusion to this theorem is well-known to hold if G is amenable. By a **linear group** we mean here a subgroup of $GL_n(F)$ for some $n < \infty$ and some field F. By [Ma40], every linear group is residually finite and hence sofic. By [Ti72] every finitely generated linear group is either virtually solvable (and hence amenable) or contains a nonabelian free group. Hence the above results imply:

Corollary 1.5. *Let G be a finitely generated infinite linear group. If $(K_1, \kappa_1), (K_2, \kappa_2)$ are probability spaces then (K_1^G, κ_1^G) is isomorphic to (K_2^G, κ_2^G) if and only if $H(\kappa_1) = H(\kappa_2)$. G is nonamenable if and only if all pairs of nontrivial Bernoulli shifts over G are weakly isomorphic. If $H(\kappa_1) = +\infty$ then there is no finite generating partition for (K_1^G, κ_1^G).*

1.1 Applications to orbit equivalence and von Neumann equivalence

In recent years, many striking new orbit equivalence rigidity and von Neumann equivalence rigidity results have been proven. For surveys, see [Sh05] and [Po07]. These results can be combined with the above theorem and corollary to obtain a classification of Bernoulli systems up to orbit equivalence or up to von Neumann equivalence for special classes of groups. Each corollary below is obtained immediately from the results above and rigidity results in [Po06] or [Po08], which apparently give the strongest rigidity results to date for Bernoulli shift actions. Let us recall the definitions.

Let G_1, G_2 be countable groups. For $i = 1, 2$, let (X_i, μ_i) be a G_i-system.

Definition 1. (X_1, μ_1) and (X_2, μ_2) are **orbit-equivalent** if there exists conull sets $X_1' \subset X_1$, $X_2' \subset X_2$ and a measure-space isomorphism $\phi : X_1' \to X_2'$ such that for all $x \in X_1', g_1 \in G_1$ there exists a $g_2 \in G_2$ such that $\phi(g_1 x) = g_2 \phi(x)$.

Definition 2. (X_1, μ_1) and (X_2, μ_2) are **von Neumann equivalent** if their crossed product von Neumann algebras are isomorphic.

It is well-known that isomorphism implies orbit-equivalence which implies von Neumann equivalence. But the converses are known to be false in general.

Definition 3. A group G is **ICC** if for every $g \in G - \{e\}$, the conjugacy class of g is infinite. ICC stands for "infinite conjugacy classes".

For more information about the following definitions, the reader is referred to [Po06].

Definition 4. A group G is w**-rigid** if it contains an infinite normal subgroup with the relative property T of Kazhdan-Margulis (in other words, (G, H) is a property (T)-pair, see [Ma82], [dHv89]). For example, all infinite groups with property T are w-rigid.

Definition 5. A subgroup $H < G$ is wq**-normal** if for every intermediate subgroup $H < H' < G$ with $H' \neq G$, there exists an element $g \in G$ such that $|gH'g \cap H'| = +\infty$. wq-normal stands for "weakly-quasinormal".

Definition 6. A group G is in the class $w\mathcal{T}_0$ if it contains a subgroup H such that

- (G, H) is a property (T) pair,
- H is not virtually abelian and
- H is wq-normal in G.

Corollary 1.6. *Let G be an ICC sofic group. Suppose one of the following conditions hold:*

1. *G is w-rigid or in the class $w\mathcal{T}_0$.*
2. *There is a nonamenable subgroup $H < G$ such that $C(H)$, the centralizer of H, is wq-normal in G and is not virtually abelian.*

Let $(K_1, \kappa_1), (K_2, \kappa_2)$ be two probability spaces such that K_1 and K_2 are finite. If (K_1^G, κ_1^G) is von Neumann equivalent to (K_2^G, κ_2^G) then $H(\kappa_1) = H(\kappa_2)$. If G is also Ornstein then the finiteness condition on K_1 and K_2 can be removed.

Proof. This follows immediately from theorem 1.1, corollary 1.2, [Po06, corollary 0.2] (if condition (1.) holds), and [Po08, theorem 1.5] (if condition (2.) holds). □

Corollary 1.7. *Let G be a sofic group satisfying: G has no nontrivial finite normal subgroups, G contains infinite commuting subgroups H, H' with H nonamenable and $H' < G$ wq-normal. Let $(K_1, \kappa_1), (K_2, \kappa_2)$ be two probability spaces such that K_1 and K_2 are finite. If (K_1^G, κ_1^G) is orbit-equivalent to (K_2^G, κ_2^G) then $H(\kappa_1) = H(\kappa_2)$. If G is also Ornstein then the finiteness condition on K_1 and K_2 can be removed.*

Proof. This follows immediately from theorem 1.1, corollary 1.2 and [Po08, corollary 1.3]. □

1.2 A brief overview of the proofs

Let G denote a countable group. Let S be a symmetric generating set for G. Symmetric means that for all $s \in S$, $s^{-1} \in S$, too.

Definition 7. A **finite S-graph** is a finite set Z with a collection $\mathbf{T} = \{T^s\}_{s \in S}$ where each T^s is a bijection $T^s : \mathrm{Dom}(T^s) \to \mathrm{Rng}(T^s)$ with $\mathrm{Dom}(T^s) \subset Z$ and $\mathrm{Rng}(T^s) \subset Z$. It is also required that $T^{s^{-1}} = (T^s)^{-1}$. In particular, $\mathrm{Dom}(T^{s^{-1}}) = \mathrm{Rng}(T^s)$ for all $s \in S$.

To simplify notation, if $z \in Z$, $s \in S$ and $T^s(z)$ is defined then we define $sz = T^s(z)$, $\mathrm{Dom}(s) = \mathrm{Dom}(T^s)$ and $\mathrm{Rng}(s) = \mathrm{Rng}(T^s)$.

We will denote an S-graph by a set Z, leaving the collection $\mathbf{T}$ implicit.

Example 1. Let $H < G$ be a subgroup and let $F \subset G/H$ be finite. For each $s \in S$, let $T^s : F \cap s^{-1}F \to F$ be the map $T^s(gH) = sgH$. This gives the structure of an S-graph to F. Two interesting examples to keep in mind are (1) when $F \subset G$ is a Fölner set (if G is amenable) and (2) when $F = G/H$ (if H has finite index in G).

For each sequence $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ of finite S-graphs such that $|Z_i| \to \infty$ as $i \to \infty$ and every G-system (X, μ) that admits a finite generating partition (definition 16) we will associate a number $f(\mathbf{Z}, X, \mu) \in \{-\infty\} \cup [0, \infty)$ such that if (Y, ν) is a G-system isomorphic to (X, μ) then $f(\mathbf{Z}, X, \mu) = f(\mathbf{Z}, Y, \nu)$. These are the new invariants.

If G is a sofic group then there exists a special type of sequence $\mathbf{Z}$ called an approximating sequence. The definition is given in section §7 where it is shown that if $\mathbf{Z}$ is an approximating sequence, K is a finite set and κ is a probability measure on K then $f(\mathbf{Z}, K^G, \kappa^G) = H(\kappa)$. This implies theorem 1.1 for finitely generated groups.

In section 8, it is shown that if (K, κ) is any probability space with K finite, (X, μ) any G-system and $\mathbf{Z}$ any approximating sequence, then $f(\mathbf{Z}, X \times K^G, \mu \times \kappa^G) = f(\mathbf{Z}, X, \mu) + f(\mathbf{Z}, K^G, \kappa^G)$. Using this, it is shown that if G is Ornstein and (L, λ) is a probability space with $H(\lambda) = \infty$ then either (L^G, λ^G) does not have a finite generating partition or $f(\mathbf{Z}, L^G, \lambda^G) = -\infty$. With this in hand, the proof of corollary 1.2 is finished for finitely generated groups.

In section 9, the case of nonfinitely generated groups is handled as a limit of the case of finitely generated groups. The proofs of theorem 1.1, corollary 1.2 and theorem 1.4 are completed.

In the section 10, the definition of $f(\mathbf{Z}, \cdot)$ is generalized to allow $\mathbf{Z}$ to be a sequence of collection of finite S-graphs. All of the results of this paper hold for this more general class of invariants. In future work, I intend to show that the f-invariant defined in [Bo08] is one of these invariants.

1.3 A very short history of the isomorphism problem

The first nontrivial properties known to be invariant under isomorphism were mixing properties. However, these were insufficient to distinguish between any two Bernoulli systems even in the classical case $G = \mathbb{Z}$. In 1958-9, A. N. Kolmogorov [Ko58, Ko59] introduced the entropy of a $\mathbb{Z}$-system. See [Ka07] for a historical survey. In 1970, D. Ornstein [Or70a, Or70b] proved that if two Bernoulli systems over $\mathbb{Z}$ have the same entropy then they are isomorphic. In 1975, A. M. Stepin showed that any group that contains a subgroup satisfying Ornstein's theorem, satisfies Ornstein's theorem itself. In 1987, D. Ornstein and B. Weiss [OW87] extended Ornstein's theorem (and many other classical entropy-theory results) to a large class of amenable groups including every countable amenable group. In recent work [Bo08], I extended this theorem to finitely generated free groups.

Organization

In §2 the new invariants are defined precisely and the main theorem, which posits their invariance, is stated. Invariance is proven in sections §3 - §5. The idea is to study two methods for obtaining a partition from a given one: by taking limits in the space of partitions and by splittings. In §3 it is proven that certain functions related to the new invariants possess a semi-continuity property on the space of partitions. In §4, it is proven that the same functions are monotone decreasing under splittings. In §5 the results from the previous two sections are combined with results in [Bo08] to finish the proof of invariance. In §6 a lower bound for the f-value of a factor is obtained. This basic result is used in subsequent sections. In §7, sofic groups are defined and it is proven the $f(\mathbf{Z}, K^G, \kappa^G) = H(\kappa)$ when G is sofic, $\mathbf{Z}$ is an approximating sequence and K is a finite set. This completes the proof of theorem 1.1 in the case when G is finitely generated. In §8, $f(\mathbf{Z}, X \times Y, \mu \times \nu)$ is compared with $f(\mathbf{Z}, X, \mu) + f(\mathbf{Z}, Y, \nu)$. The results obtained are used to finish the proof of corollary 1.2 in the case when G is finitely generated. In §9, nonfinitely generated group actions are studied and the proofs of theorem 1.1, corollary 1.2 and theorem 1.4 are completed. In §10, the relationship between the new invariants and the f-invariant from [Bo08] is briefly discussed.

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2 Approximations

Throughout this paper, G denotes a fixed countable group and S is a symmetric generating set for G. Symmetric means that for all $s \in S$, $s^{-1} \in S$.

Definition 8. A G**-process** is a triple (X, μ, α) where (X, μ) is a G-system and $\alpha = (A_1, ..., A_u)$ is a finite ordered measurable partition of X. The term **process** is used when the group G is understood.

Definition 9. A **finite partitioned** S**-graph** is a pair (Z, β) where Z is a finite S-graph and β is an ordered partition of Z. See definition 7 for the definition of a finite S-graph.

Definition 10. Let (X, μ, α) be a process and (Z, β) be a finite partitioned S-graph. Let $\alpha = (A_1, ..., A_u)$ and $\beta = (B_1, ..., B_v)$. For all $i > u$ and all $j > v$ define $A_i = B_j = \emptyset$. Let $\epsilon > 0$. Let ζ be the uniform probability measure on Z. Then (Z, β) is a **finite** ϵ**-approximation** to (X, μ, α) if

$$\sum_{s\in S}\sum_{i,j=1}^{\infty}\Big|\mu(A_i \cap sA_j) - \zeta\big(B_i \cap s(B_j \cap \mathrm{Dom}(s))\big)\Big| \le \epsilon.$$

Definition 11. For a partition α of X, let $|\alpha|$ denote the number of atoms $A \in \alpha$ with $\mu(A) > 0$. Similarly, if β is a partition of Z then let $|\beta|$ denote the number of atoms of nonempty atoms of β.

Definition 12. Let (X, μ, α) be a process and let Z be a finite S-graph. For $\epsilon, c > 0$ let $\mathcal{FA}_c(Z, \alpha, \epsilon)$ be the set of all ordered partitions β of Z such that (Z, β) is a finite ϵ-approximation of (X, μ, α) and $|\beta| \le |\alpha| + c$. Write $\mathcal{FA}(Z, \alpha, \epsilon) = \mathcal{FA}_0(Z, \alpha, \epsilon)$.

Definition 13. Let (X, μ, α) be a process and let $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ be a sequence of finite S-graphs. For $\epsilon > 0$, define

$$F_c(\mathbf{Z}, \alpha, \epsilon) := \limsup_{i\to\infty} \frac{1}{|Z_i|} \log |\mathcal{FA}_c(Z_i, \alpha, \epsilon)|.$$

When the subscript c is omitted, it is assumed to equal 0. Thus, $\mathcal{FA}(\mathbf{Z}, \alpha, \epsilon) = \mathcal{FA}_0(\mathbf{Z}, \alpha, \epsilon)$. Here we are using the convention $\log(0) = -\infty$. Although it is not conventional to do so in this context, we will assume that the base of logarithm above is e instead of 2. This will make certain arguments involving Stirling's approximation slightly simpler.

If $0 < \epsilon_1 \le \epsilon_2$, then $\mathcal{FA}_c(Z_i, \alpha, \epsilon_1) \subset \mathcal{FA}_c(Z_i, \alpha, \epsilon_2)$. Therefore, $F_c(\mathbf{Z}, \alpha, \epsilon_1) \le F_c(\mathbf{Z}, \alpha, \epsilon_2)$. So we define

$$F_c(\mathbf{Z}, \alpha) = \lim_{\epsilon\to 0} F_c(\mathbf{Z}, \alpha, \epsilon) = \inf_{\epsilon>0} F_c(\mathbf{Z}, \alpha, \epsilon).$$

An easy calculation (lemma 2.1 below) shows that if $|Z_i| \to \infty$ as $i \to \infty$ then the value of $F_c(\mathbf{Z}, \alpha)$ does not depend on c. Thus we write $F(\mathbf{Z}, \alpha) = F_c(\mathbf{Z}, \alpha)$ for any $c \geq 0$.

Observe that $F_c(\mathbf{Z}, \alpha, \epsilon)$ and $F(\mathbf{Z}, \alpha)$ does not depend on the ordering of α. So it is well-defined for unordered partitions α.

Lemma 2.1. *Let $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ be a sequence of finite S-graphs such that $|Z_i| \to \infty$ as $i \to \infty$. Let (X, μ) be a G-system. Then for any $c \geq 0$,*

$$F_0(\mathbf{Z}, \alpha) = F_c(\mathbf{Z}, \alpha).$$

Proof. For any i and any $\epsilon > 0$, $\mathcal{FA}_0(Z_i, \alpha, \epsilon) \subset \mathcal{FA}_c(Z_i, \alpha, \epsilon)$. Hence $F_0(\mathbf{Z}, \alpha) \leq F_c(\mathbf{Z}, \alpha)$.

For the converse, fix a finite S-graph Z with uniform probability measure ζ and define $\Phi : \mathcal{FA}_c(Z, \alpha, \epsilon) \to \mathcal{FA}_0(Z, \alpha, \epsilon)$ as follows. Let $u = |\alpha|$ be the number of atoms in α. If $\beta = (B_1, ..., B_v) \in \mathcal{FA}_c(Z, \alpha, \epsilon)$ then set $\Phi(\beta) = \gamma = (C_1, ..., C_u)$ where $C_j = B_j$ for all $1 \leq j < u$ and $C_u = \bigcup_{i=u}^{v} B_i$. An easy calculation shows that $\Phi(\beta) \in \mathcal{FA}_0(Z, \alpha, \epsilon)$ as claimed.

In order to obtain a lower bound on $|\mathcal{FA}_0(Z, \alpha, \epsilon)|$ we estimate the number of Φ-preimages of a given element $\gamma = (C_1, ..., C_u) \in \mathcal{FA}_0(Z, \alpha, \epsilon)$.

Let J be the set of all functions $j : \mathrm{Dom}(j) \to \{u, u+1, .., u+c\}$ where $\mathrm{Dom}(j) \subset C_u$ has cardinality $m = \min(|C_u|, \lceil \epsilon |Z| \rceil)$. For each $j \in J$, let $\beta^j = (B_1^j, ..., B_{u+c}^j)$ where $B_i^j = C_i$ for $1 \leq i < u$, $B_u^j = (C_u - \mathrm{Dom}(j)) \cup j^{-1}(u)$ and $B_i^j = j^{-1}(i)$ for $u < i \leq v$.

Because the domain of Φ is contained in $|\mathcal{FA}_c(Z, \alpha, \epsilon)|$, it follows that every element of $\Phi^{-1}(\gamma)$ is of the form β^j for some $j \in J$. Thus,

$$|\Phi^{-1}(\gamma)| \quad \leq \quad |J| = \binom{|C_u|}{m} (c+1)^m \leq \binom{|Z|}{m} (c+1)^m.$$

By Stirling's approximation, if $|Z|$ is sufficiently large,

$$|\Phi^{-1}(\gamma)| \leq \exp\Big(\big(H(\epsilon) + \epsilon\big)|Z| \Big) (c+1)^{\epsilon|Z|+1}$$

where $H(\epsilon) = -\epsilon \log(\epsilon) - (1-\epsilon)\log(1-\epsilon)$. Thus,

$$|\mathcal{FA}_0(Z, \alpha, \epsilon)| \geq \exp\Big(-\big(H(\epsilon) + \epsilon\big)|Z| \Big) (c+1)^{-\epsilon|Z|-1} |\mathcal{FA}_c(Z, \alpha, \epsilon)|.$$

Since $|Z_i| \to \infty$ as $i \to \infty$, this implies

$$F_0(Z, \alpha, \epsilon) \geq F_c(Z, \alpha, \epsilon) - \big(H(\epsilon) + \epsilon + \epsilon \log(c+1)\big).$$

Let $\epsilon \to 0$ to obtain $F_0(Z, \alpha) \geq F_c(Z, \alpha)$. $\square$

Definition 14. The **word-metric on G induced by** S is defined as follows. For $g \in G$, let $|g|$ be the smallest number n such that there exists elements $s_i \in S$ with $g = s_1 s_2 ... s_n$. For $g_1, g_2 \in G$, let $d(g_1, g_2) = |g_1 g_2^{-1}|$. Let $B(g, n) = \{g' \in G \mid d(g', g) \leq n\}$.

Definition 15. If α, β are two partitions of X then the **join** of α and β is the partition

$$\alpha \vee \beta := \{A \cap B \mid A \in \alpha, B \in \beta\}.$$

Also define

$$\alpha^n := \bigvee_{g \in B(e,n)} g\alpha.$$

Definition 16. Let (X, μ) be a G-system. Let α be a partition. Let Σ_α be the smallest G-invariant σ-algebra containing the atoms of α. Then α is **generating** if for all measurable sets $A \subset X$ there exists a set $A' \in \Sigma_\alpha$ such that $\mu(A \Delta A') = 0$.

The main theorem of this paper is:

Theorem 2.2. *Let G be a countable group and let S be a symmetric generating set for G. Let (X, μ, α) be a G-process. Let $\mathbf{Z} = \{Z_i\}_{i=1}^\infty$ be a sequence of finite S-graphs such that $|Z_i| \to \infty$ as $i \to \infty$. Then the following limit exists:*

$$f(\mathbf{Z}, \alpha) = \lim_{n \to \infty} F(\mathbf{Z}, \alpha^n).$$

Moreover, $f(\mathbf{Z}, \alpha) = \inf_{n \geq 0} F(\mathbf{Z}, \alpha^n)$. If α and β are two finite generating partitions for (X, μ) then $f(\mathbf{Z}, \alpha) = f(\mathbf{Z}, \beta)$. Therefore, we may define $f(\mathbf{Z}, X, \mu) = f(\mathbf{Z}, \alpha)$ for any generating partition α. The number $f(\mathbf{Z}, X, \mu)$ depends on (X, μ) only up to isomorphism.

3 Upper Semi-Continuity

Definition 17. Let (X, μ) be a G-system. Let $\mathcal{P}_o$ denote the set of all finite measurable ordered partitions of X. Given $\alpha = (A_1, .., A_u), \beta = (B_1, ..., B_v) \in \mathcal{P}_o$ define $d_o(\alpha, \beta)$ as follows. For all $i > u$ and all $j > v$ define $A_i = B_j = \emptyset$. Then

$$d_o(\alpha, \beta) = \sum_{s \in S} \sum_{i,j=1}^{\infty} |\mu(A_i \cap sA_j) - \mu(B_i \cap sB_j)|.$$

This defines a pseudo-metric on $\mathcal{P}_o$.

Definition 18. A function $F : \mathcal{P}_o \to \mathbb{R} \cup \{-\infty\}$ is **weakly upper semi-continuous** if the following is true. Suppose α is a partition in $\mathcal{P}_o$ and $\{\alpha_i\}_{i=1}^\infty$ is a sequence of ordered partitions such that $\lim_{i \to \infty} d_o(\alpha_i, \alpha) = 0$ and for some $c \geq 0$, $|\alpha_i| \leq |\alpha| + c$ for every $i \geq 0$. Then

$$\limsup_{i \to \infty} F(\alpha_i) \leq F(\alpha).$$

Proposition 3.1. *Let (X, μ) be a G-system and let $\mathbf{Z} = \{Z_i\}_{i=1}^\infty$ be a sequence of finite S-graphs. Then the function $F(\mathbf{Z}, \cdot) : \mathcal{P}_o \to \mathbb{R} \cup \{-\infty\}$ is weakly upper semi-continuous.*

Proof. Let $\alpha = (A_1, ..., A_u)$. If Z is any finite S-graph and $\beta \in \mathcal{P}_o$ is an ordered partition $|\beta| \le |\alpha| + c$ and $d_o(\alpha, \beta) < \delta$ then for any $\epsilon > 0$,

$$\mathcal{FA}(Z, \beta, \epsilon) \subset \mathcal{FA}_c(Z, \alpha, \epsilon + \delta).$$

This follows directly from the definitions of d_o and $\mathcal{FA}$. So if α_i is as in the definition above, $\epsilon_i = d_o(\alpha, \alpha_i)$ and $j > 0$ is arbitrary then

$$|\mathcal{FA}(Z_j, \alpha_i, \epsilon)| \le |\mathcal{FA}_c(Z_j, \alpha, \epsilon + \epsilon_i)|.$$

By taking the logarithm, dividing by $|Z_j|$ and taking the limit supremum with respect to j we obtain, $F(\mathbf{Z}, \alpha_i, \epsilon) \le F_c(\mathbf{Z}, \alpha, \epsilon + \epsilon_i)$ for all i. Now take a limit as $\epsilon \to 0$ to obtain $F(\mathbf{Z}, \alpha_i) \le F_c(\mathbf{Z}, \alpha, \epsilon_i)$. Finally, take the limsup as i tends to infinity to obtain $\limsup_{i \to \infty} F(\mathbf{Z}, \alpha_i) \le F(\mathbf{Z}, \alpha)$. □

4 Splittings

Definition 19. Let α, β be a finite partitions of X. We say α **refines** β if for all $A \in \alpha$ there exists a $B \in \beta$ such that $A \subset B$. In this case, β is a **coarsening** of α. To express this, we write $\alpha \ge \beta$.

Definition 20. Let α be a finite partition of X. A **simple splitting** of α is a partition σ such that $\alpha \vee t\alpha \ge \sigma \ge \alpha$ for some $t \in S$. A partition γ is a **splitting** of α if it can be obtained from α by a sequence of simple splittings. This means that there exists a sequence $\alpha = \sigma_1, \sigma_2, ..., \sigma_n = \gamma$ such that for all $1 \le i < n$, σ_{i+1} is a simple splitting of σ_i.

This definition is slightly more general than the one given in [Bo08, section 4]. Either definition could be used in either paper without changing any results.

Proposition 4.1. *If α is a splitting of a partition β and $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ is a sequence of finite S-graphs with $|Z_i| \to \infty$ as $i \to \infty$ then $F(\mathbf{Z}, \alpha) \le F(\mathbf{Z}, \beta)$.*

Proof. It suffices to prove the proposition in the special case in which α is a simple splitting of β. So assume that, for some $t \in S$, $\beta \vee t\beta \ge \alpha \ge \beta$. Let $\beta = (B_1, ..., B_v)$ and $\alpha = (A_1, .., A_u)$. Let Z be a finite S-graph with uniform probability measure ζ. Let $\epsilon > 0$.

Let $\phi : \{1, ..., u\} \to \{1, ..., v\}$ be the map $\phi(i) = j$ if $A_i \subset B_j$. Let $\psi : \{1, .., v\} \times \{1, ..., v\} \to \{1, .., u\}$ be the map $\psi(i, j) = k$ if $B_i \cap tB_j \subset A_k$. ψ is not always well-defined because it can happen that $B_i \cap tB_j = \emptyset$. But if this occurs, then define $\psi(i, j)$ arbitrarily.

Define the **coarsening map** $\Phi : \mathcal{FA}(Z, \alpha, \epsilon) \to \mathcal{FA}(Z, \beta, \epsilon)$ as follows. If $\bar{\alpha} = (\bar{A}_1, ..., \bar{A}_u) \in \mathcal{FA}(Z, \alpha, \epsilon)$ then let $\Phi(\bar{\alpha}) = \bar{\beta} = (\bar{B}_1, ..., \bar{B}_v)$ where for each j,

$$\bar{B}_j = \bigcup_{\{i : \phi(i) = j\}} \bar{A}_i.$$

The proposition would follow almost immediately if it were true that Φ is injective. This need not be the case. However, each partition $\bar{\beta}$ in the image of Φ has a relatively small number of preimages. The next task is to obtain an upper bound on this number.

Now fix $\bar{\beta} = (\bar{B}_1, ..., \bar{B}_v) \in \mathcal{FA}(Z, \beta, \epsilon)$. Define $\bar{\alpha} = (\bar{A}_1, ..., \bar{A}_{u+1})$ by:

$$\bar{A}_i = \bigcup_{\{j,k:\psi(j,k)=i\}} \bar{B}_j \cap t(\bar{B}_k \cap \mathrm{Dom}(t)),$$

$$\bar{A}_{u+1} = Z - \cup_{i=1}^{u} \bar{A}_i = Z - \mathrm{Dom}(t^{-1}).$$

Let $\bar{\alpha}' = (\bar{A}'_1, ..., \bar{A}'_u)$ be any partition in $\Phi^{-1}(\bar{\beta})$. We will show that $\bar{\alpha}'$ equals $\bar{\alpha}$ on a large subset of Z. For $i > u+1$ and $j > u$ define $\bar{A}_i = \bar{A}'_j = \emptyset$. Let Z' be the set of all $z \in \mathrm{Dom}(t^{-1}) \subset Z$ such that if i and j are such that $z \in \bar{A}'_i \cap t(\bar{A}'_j \cap \mathrm{Dom}(t))$ then $A_i \cap tA_j \neq \emptyset$.

Claim 1: $\zeta(Z') \geq 1 - \epsilon$.

Since $\bar{\alpha}' \in \mathcal{FA}(Z, \alpha, \epsilon)$,

$$\sum_{s\in S} \sum_{i,j=1}^{\infty} \Big| \mu(A_i \cap sA_j) - \zeta\big(\bar{A}'_i \cap s(\bar{A}'_j \cap \mathrm{Dom}(s))\big) \Big| \leq \epsilon.$$

Let K be the set of (i,j) such that $A_i \cap tA_j \neq \emptyset$. The inequality above implies

$$\begin{aligned} \epsilon &\geq \sum_{(i,j)\in K} \Big| \mu(A_i \cap tA_j) - \zeta\big(\bar{A}'_i \cap t(\bar{A}'_j \cap \mathrm{Dom}(t))\big) \Big| \\ &= \sum_{(i,j)\in K} \Big| \mu(A_i \cap tA_j) - \zeta\big(\bar{A}'_i \cap t(\bar{A}'_j \cap \mathrm{Dom}(t)) \cap Z'\big) \Big|. \end{aligned}$$

From this it follows that

$$\zeta(Z') = \sum_{(i,j)\in K} \zeta\big(\bar{A}'_i \cap t(\bar{A}'_j \cap \mathrm{Dom}(t)) \cap Z'\big) \geq -\epsilon + \sum_{(i,j)\in K} \mu(A_i \cap tA_j) = 1 - \epsilon.$$

Claim 2: $\bigcup_{i=1}^{\infty} \bar{A}'_i \Delta \bar{A}_i \subset Z - Z'$.

The definition of Z' implies $Z - \mathrm{Dom}(t^{-1}) \subset Z - Z'$. In particular $\bar{A}_{u+1} \subset Z - Z'$. So fix $z \in \mathrm{Dom}(t^{-1}) \cap \bigcup_{i=1}^{\infty} \bar{A}'_i \Delta \bar{A}_i$ and let i, j, k, l be such that

$$z \in \bar{A}'_i \cap t(\bar{A}'_j \cap \mathrm{Dom}(t)) \cap \bar{A}_k \cap t(\bar{A}_l \cap \mathrm{Dom}(t)).$$

By definition of the coarsening map,

$$z \in \bar{B}_{\phi(i)} \cap t(\bar{B}_{\phi(j)} \cap \mathrm{Dom}(t)) \subset \bar{A}_{\psi(\phi(i),\phi(j))}.$$

Since $z \in \bar{A}_k$, it follows that $\psi(\phi(i), \phi(j)) = k$. So,

$$A_i \cap tA_j \subset B_{\phi(i)} \cap tB_{\phi(j)} \subset A_{\psi(\phi(i),\phi(j))} = A_k.$$

Since $i \neq k$, this implies $A_i \cap tA_j = \emptyset$ which implies $z \in Z - Z'$ as claimed.

Claims 1 and 2 imply that every $\bar{\alpha}' \in \Phi^{-1}(\bar{\beta})$ is equal to $\bar{\alpha}$ up to a set of ζ-measure ϵ. Now consider the set Q of all functions $q : \mathrm{Dom}(q) \to \{1, ..., u\}$ where $\mathrm{Dom}(q) \subset Z$ is a set with cardinality $|\mathrm{Dom}(q)| = \lceil \epsilon |Z| \rceil$. For each element $q \in Q$, obtain an ordered partition $\bar\alpha^q = (\bar A_1^q, ..., \bar A_u^q, \bar A_{u+1}^q)$ by defining

$$\bar A_i^q = \big(\bar A_i - \mathrm{Dom}(q)\big) \cup q^{-1}(i).$$

The above claims imply that for every partition $\bar\alpha' \in \Phi^{-1}(\bar\beta)$ there is a $q \in Q$ such that $\bar\alpha' = \bar\alpha^q$. Therefore,

$$|\Phi^{-1}(\bar\beta)| \quad \le \quad |Q| = \binom{|Z|}{\lceil \epsilon |Z| \rceil} u^{\lceil \epsilon |Z| \rceil}.$$

It follows from Stirling's approximation that when $|Z|$ is sufficiently large,

$$\binom{|Z|}{\lceil \epsilon |Z| \rceil} \quad \le \quad e^{[H(\epsilon)+\epsilon]|Z|}$$

where $H(\epsilon) = -\epsilon \log(\epsilon) - (1-\epsilon)\log(1-\epsilon)$. Thus,

$$|\Phi^{-1}(\bar\beta)| \le \exp\Big([H(\epsilon)+\epsilon]|Z| + (\epsilon|Z|+1)\log(u)\Big).$$

So when i is sufficiently large,

$$\Big|\mathcal{FA}(Z_i, \alpha, \epsilon)\Big| \le \exp\Big([H(\epsilon)+\epsilon]|Z_i| + (\epsilon|Z_i|+1)\log(u)\Big)\Big|\mathcal{FA}(Z_i, \beta, \epsilon)\Big|.$$

This implies $F(\mathbf{Z}, \alpha) \le F(\mathbf{Z}, \beta)$ as claimed. □

5 The proof of the theorem 2.2

Theorem 5.1. *Let $F : \mathcal{P}_o \to \mathbb{R}$ be any weakly upper semi-continuous function. Suppose that F is monotone decreasing under splittings. I.e., whenever σ is a splitting of a partition α then $F(\sigma) \le F(\alpha)$. Define $f : \mathcal{P}_o \to \mathbb{R}$ by*

$$f(\alpha) = \lim_{n\to\infty} F(\alpha^n) = \inf_n F(\alpha^n).$$

Then, for any two generating partitions α_1 and α_2, $f(\alpha_1) = f(\alpha_2)$. So we may define $f(X, \mu) = f(\alpha)$ for any generating partition α. Then $f(X, \mu)$ is an isomorphism invariant of the system (X, μ), i.e., if (Y, ν) is any system that is isomorphic to (X, μ) then $f(X, \mu) = f(Y, \nu)$.

Proof. This theorem is proven in [Bo08, theorem 4.4]. Actually, it was assumed there that F is continuous; but the proof requires only weak upper semi-continuity. This is because the proof of theorem 3.4 of [Bo08] gives a stronger result than stated. It proves (without any additional effort) that if α, β are any two finite ordered generating partitions of X then there is a sequence $\{\alpha_i\}_{i=1}^\infty$ of ordered partitions such that each α_i is topologically equivalent to α, $|\alpha_i| \le |\beta| + |\beta|^2$ for all i and $\lim_{i\to\infty} d_o(\alpha_i, \beta_i) = 0$. □

Proof of theorem 2.2. Fix a sequence $\mathbf{Z} = \{(Z_i, \zeta_i)\}_{i=1}^\infty$ of finite partial (G, S)-actions. For any partition α, let $F(\alpha) = F(\mathbf{Z}, \alpha)$. By proposition 3.1 and proposition 4.1, the hypothesis of theorem 5.1 are satisfied. The conclusion of theorem 5.1 finishes the proof. □

6 A lower bound for the f-value of a factor

The next lemma and its corollary are fundamental results that are used in theorem 7.2 below to bound the $f(\mathbf{Z}, \cdot)$-value of a Bernoulli system.

Lemma 6.1. *Let G be a countable group, S a finite symmetric generating set for G, $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ a sequence of S-graphs with $|Z_i| \to \infty$ as $i \to \infty$ and (X, μ) a G-system. Let α and β be two finite ordered partitions of X and suppose that α refines β. Then*

$$F(\mathbf{Z}, \beta) - H(\beta) \geq F(\mathbf{Z}, \alpha) - H(\alpha).$$

Proof. Let Z be an S-graph. Let $\beta = (B_1, ..., B_v)$. For each $1 \leq i \leq v$ let $\{A_{i,1}, ..., A_{i,k_i}\}$ be the collection of atoms of α contained in B_i. So any partition $\bar{\alpha} \in \mathcal{FA}(Z, \alpha, \epsilon)$ can be written as $\bar{\alpha} = \{\bar{A}_{i,j} \mid 1 \leq i \leq v, 1 \leq j \leq k_i\}$ such that

$$\sum_{s \in S} \sum_{i_1, i_2, j_1, j_2 = 1}^{\infty} \Big| \mu(A_{i_1,j_1} \cap sA_{i_2,j_2}) - \zeta\big(\bar{A}_{i_1,j_1} \cap s(\bar{A}_{i_2,j_2} \cap \mathrm{Dom}(s))\big) \Big| \leq \epsilon$$

where $A_{i,j} = \bar{A}_{i,j} = \emptyset$ whenever $i > v$ or $j > k_i$.

Let $\mathcal{M}$ be the set of all $v \times u$ matrices. Let $\Psi : \mathcal{FA}(Z, \alpha, \epsilon) \to \mathcal{M}$ be defined by

$$\Psi(\bar{\alpha})_{ij} = \frac{|\bar{A}_{i,j}|}{|Z|}$$

where $\bar{\alpha} = (\bar{A}_{i,j})$.

If M is in the image of Ψ then for each i, j, the denominator of $M_{i,j}$ is at most $|Z|$. Hence the number of matrices in the images of Ψ is bounded:

$$|\Psi(\mathcal{FA}(Z, \alpha, \epsilon))| \leq |Z|^u$$

where $u = |\alpha|$ is the number of atoms in α.

For $\epsilon > 0$, let $\Phi : \mathcal{FA}(Z, \alpha, \epsilon) \to \mathcal{FA}(Z, \beta, \epsilon)$ be the coarsening map. That is, $\Phi(\bar{\alpha}) = \bar{\beta} = (\bar{B}_1, .., \bar{B}_v)$ where $\bar{B}_i = \cup_j \bar{A}_{i,j}$. For any $M \in \mathcal{M}$,

$$|\Phi^{-1}(\bar{\beta}) \cap \Psi^{-1}(M)| \leq \prod_{i=1}^{v} \binom{|\bar{B}_i|}{M_{i,1}|Z|, M_{i,2}|Z|, ..., M_{i,k_i}|Z|}.$$

Indeed, if Φ were defined from $\mathcal{FA}(Z, \alpha, \infty)$ instead then the above would be an equality. Let

$$H(M) = -\sum_{i,j} M_{i,j} \log(M_{i,j}).$$

By Stirling's approximation, if $|Z|$ is sufficiently large then

$$|\Phi^{-1}(\bar{\beta}) \cap \Psi^{-1}(M)| \leq \exp\Big(|Z|(H(M) - H(\bar{\beta}) + \epsilon)\Big).$$

Since $H(\cdot)$ is continuous on the space of partitions, it follows that there is some function j depending only on $H(\alpha)$ and $H(\beta)$ such that for all M in the image of Ψ and all $\bar{\beta} \in \mathcal{FA}(\mathbf{Z}, \beta, \epsilon)$,

$$|H(\alpha) - H(M)| + |H(\bar{\beta}) - H(\beta)| < j(\epsilon)$$

and $\lim_{\epsilon \to 0} j(\epsilon) = 0$. Thus,

$$\begin{aligned} |\Phi^{-1}(\bar{\beta})| &\leq \sum_{M \in \Psi(\mathcal{FA}(Z,\alpha,\epsilon))} |\Phi^{-1}(\bar{\beta}) \cap M| \\ &\leq |Z|^u \exp\Big(|Z|(H(\alpha) - H(\beta) + \epsilon + j(\epsilon))\Big). \end{aligned}$$

Therefore,

$$|\mathcal{FA}(Z, \alpha, \epsilon)| \leq |Z|^u \exp\Big(|Z|(H(\alpha) - H(\beta) + \epsilon + j(\epsilon))\Big) |\mathcal{FA}(Z, \beta, \epsilon)|.$$

This implies $F(\mathbf{Z}, \alpha) \leq F(\mathbf{Z}, \beta) + H(\alpha) - H(\beta)$ as claimed. □

Corollary 6.2. *Let $(X, \mu), (Y, \nu)$ be G-systems with finite generating partitions α, β respectively. If (X, μ) factors onto (Y, ν) and $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ is any sequence of S-graphs with $|Z_i| \to \infty$ as $i \to \infty$ then*

$$f(\mathbf{Z}, Y, \nu) \geq f(\mathbf{Z}, X, \mu) - H(\alpha).$$

Proof. Without loss of generality, we can assume that $X = Y$ and ν equals μ restricted to the σ-algebra generated by β and the action of G. By the previous lemma for each $n \geq 0$,

$$F(\mathbf{Z}, \beta^n) - H(\beta^n) \geq F(\mathbf{Z}, \alpha \vee \beta^n) - H(\alpha \vee \beta^n).$$

Since $H(\alpha \vee \beta^n) \leq H(\alpha) + H(\beta^n)$ this implies

$$F(\mathbf{Z}, \beta^n) \geq F(\mathbf{Z}, \alpha \vee \beta^n) - H(\alpha).$$

Since F is monotone decreasing under splittings (by proposition 4.1) $f(\mathbf{Z}, \alpha) = f(\mathbf{Z}, \alpha \vee \beta^n) \leq F(\mathbf{Z}, \alpha \vee \beta^n)$ for all $n \geq 0$. Thus, by taking limits as $n \to \infty$ we obtain

$$f(\mathbf{Z}, Y, \nu) = f(\mathbf{Z}, \beta) \geq f(\mathbf{Z}, \alpha) - H(\alpha) = f(\mathbf{Z}, X, \mu) - H(\alpha).$$

□

7 Sofic Groups

Definition 21. Let Z be a finite S-graph. For each $r > 0$, let Z_r be the set of all $z \in Z$ such that for all $s_1, ..., s_r, t_1, .., t_r \in S \cup \{e\}$, $s_1 ... s_r z$ is well-defined and

$$s_1(...(s_{r-1}(s_r z))...) = t_1(...(t_{r-1}(t_r z))...) \iff s_1 .... s_r = t_1 ... t_r.$$

Recall that $B(e, r)$ denotes the ball of radius r in the Cayley graph of G with respect to the word metric defined by S. If $g \in B(e, r)$ and $z \in Z_r$ then define $gz = s_1 ... s_r z$ where $s_1, ..., s_r \in S \cup \{e\}$ is such that $s_1 ... s_r = g$.

For $\delta > 0$, Z is a finite (r, δ)**-approximation of** (G, S) if $|Z_r| \geq (1 - \delta)|Z|$.

Definition 22. An **approximating sequence for** (G, S) is a sequence $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ such that for each i, Z_i is a finite (r_i, δ_i)-approximation of (G, S) and $\lim_{i\to\infty} r_i = +\infty$, $\lim_{i\to\infty} \delta_i = 0$.

Definition 23. (G, S) is **sofic** if there exists an approximating sequence for (G, S). It is known [We00] that if (G, S) is sofic and S' is another finite symmetric generating set for G then (G, S') is also sofic. Therefore, the group G is called **sofic** if (G, S) is sofic for some finite symmetric generating set S. It is easy to show that every finitely generated subgroup of a finitely generated sofic group is sofic. Therefore, a (possibly nonfinitely generated) group is said to be sofic if all of its finitely generated subgroups are sofic.

All residually finite groups G are sofic. This can be seen by choosing the sequence $\{Z_i\}$ to be such that each $Z_i = G/H_i$ for some finite index subgroup $H_i < G$ and $\cap_i H_i = \{1\}$. All amenable groups are sofic, too; take $\mathbf{Z}$ to be a Fölner sequence in G. For more fundamental results on sofic groups, see [ES06].

Lemma 7.1. *Let K be a finite set and let κ be a probability measure on K. Let K^m denote the m-fold Cartesian product and let κ^m be the product measure on K^m. Then for every $\epsilon > 0$, there exists an $M > 0$ such that for all $m > M$ there exists a set $Q \subset K^m$ satisfying*

- $\kappa^m(Q) > 1 - \epsilon$,
- $\forall\, q \in Q$, $e^{-mH(\kappa)-m\epsilon} \le \kappa^m\big(\{q\}\big) \le e^{-mH(\kappa)+m\epsilon}$.

Proof. This follows immediately from the Shannon-McMillan-Breiman theorem. The base equals e instead of 2 because we chose e as the base of the logarithm in the definition of $H(\kappa)$. □

Theorem 7.2. *Let S be a finite symmetric generating set for G. Let K be a finite set and let κ be a probability measure on K. Let $\mathbf{Z}$ be an approximating sequence for (G, S). Then $f(\mathbf{Z}, K^G, \kappa^G) = H(\kappa)$.*

Proof. Let $K = \{1, ..., u\}$. For $k \in K$, let $A_k = \{x \in K^G \mid x(e) = k\}$. Let $\alpha = \{A_k \mid k \in K\}$. α is a generating partition for (K^G, κ^G). So it suffices to show that $f(\mathbf{Z}, \alpha) = H(\kappa)$.

The upper bound follows from lemma 6.1. To see this, let β be the trivial partition $\beta = \{X\}$. Then $F(\mathbf{Z}, \beta) = H(\beta) = 0$. So lemma 6.1 implies $f(\mathbf{Z}, \alpha) \le F(\mathbf{Z}, \alpha) \le H(\alpha) = H(\kappa)$.

Let $\epsilon, r, \delta > 0$ and let Z be a finite (r, δ)-approximation of (G, S). Let ζ be the uniform probability measure on Z.

The idea for the lower bound is this. Every function $q : Z \to K$ induces a partition β_q whose atoms are the preimages of elements in K. If the S-graph structure on Z induces an action of the group G then the partition $\beta_q^n := \bigvee_{g\in B(e,n)} g\beta$ is well-defined. It is possible to construct an analogous partition in the general case. If n is sufficiently small relative to r, δ is sufficiently small and $|Z|$ is sufficiently large, then it can be shown that for most functions $q : Z \to K$, β_q^n is a good approximation to α^n. From this and the previous lemma, we obtain a lower bound on $|\mathcal{FA}(Z, \alpha, \epsilon)|$.

Now let Z_n be defined as in definition 21. Given an arbitrary ordered partition $\beta = (B_1, ..., B_u)$ of Z let $q_\beta : Z \to K$ be the function $q_\beta(z) = k$ if $z \in B_k$.

Recall that $B(e,n)$ denotes the ball of radius n centered at the identity element e in the group G with respect to the word metric induced by S. For each function $j : B(e,n) \to K$ there exists a unique atom $C_j \in \alpha^n$ defined by $C_j = \{x \in X \mid x(g) = j(g)\ \forall g \in B(e,n)\}$. I.e., $x \in C_j$, if $g^{-1}x \in A_{j(g)}$ for all $g \in B(e,n)$. By analogy, let $D_j = \{z \in Z_n \mid q_\beta(g^{-1}z) = j(g) \forall g \in B(e,n)\}$.

Define $\beta^n = \{D_j \mid j : B(e,n) \to K\} \cup \{Z - Z_n\}$. It is consistent with definition 10 to say that $\beta^n \in \mathcal{FA}_1(Z, \alpha^n, \epsilon)$ if and only if

$$\sum_{s\in S} \sum_{j,k:B(e,n)\to K} \Big|\mu(C_j \cap sC_k) - \zeta\big(D_j \cap s(D_k \cap \mathrm{Dom}(s))\big)\Big| \le \epsilon.$$

Fix $n > 0$ and let

$$\epsilon_2 = \epsilon |K|^{-|B(e,n)|}|S|^{-1}.$$

Observe that if $|\mu(C_j) - \zeta(D_j)| \le \epsilon_2$ for all $j : B(e,n) \to K$ then $\beta^{n-1} \in \mathcal{FA}(Z, \alpha^{n-1}, \epsilon)$. This is because the number of atoms of α^n contained in any given atom of α^{n-1} is $|K|^{|B(e,n)|-|B(e,n-1)|} \le K^{|B(e,n)|}$.

Recall that any element $q \in K^Z$ can be considered as a function $q : Z \to K$. For $q \in K^Z$, let β_q be the ordered partition $\beta_q = (B_{q,1}, ..., B_{q,u})$ where $B_{q,i} = q^{-1}(i)$. We will estimate $\kappa^Z(\{q \in K^Z \mid \beta_q^{n-1} \in \mathcal{FA}(Z, \alpha^{n-1}, \epsilon)\})$ and then use the previous lemma to obtain a lower bound on the cardinality of $\mathcal{FA}(Z, \alpha^{n-1}, \epsilon)$.

It will be convenient to use probabilistic language. So, let q be a random element of K^Z with law κ^Z. Then for any subset $Q \subset K^Z$ and any function $\phi : K^Z \to \mathbb{R}$ define

$$Pr[q \in Q] := \kappa^Z(Q),\ \mathbb{E}[\phi] := \int \phi \, d\kappa^Z,\ Var(\phi) := \mathbb{E}[\phi^2] - \mathbb{E}[\phi]^2.$$

For each function $j : B(e,n) \to K$ and each $z \in Z$ define $\phi_{j,z} : K^Z \to \mathbb{R}$ by $\phi_{j,z}(q) = 1$ if $z \in D_j$ where D_j is defined as above for $\beta = \beta_q$. In other words, $\phi_{j,z}(q) = 1$ if $z \in Z_n$ and for all $g \in B(e,n)$, $j(g) = q(g^{-1}z)$. Otherwise, set $\phi_{j,z}(q) = 0$. Let $\phi_j = \sum_{z\in Z} \phi_{j,z}$. Observe that $\frac{\phi_j(q)}{|Z|} = \zeta(D_j)$ where D_j is defined as above for $\beta = \beta_q$.

If $z \in Z_n$ then $\mathbb{E}[\phi_{j,z}] = \mu(C_j)$. If $z \notin Z_n$ then $\phi_{j,z} \equiv 0$. Thus,

$$\mathbb{E}[\phi_j] \;=\; \sum_{z\in Z} \mathbb{E}[\phi_{j,z}] = \mu(C_j)|Z_n|.$$

Let $d(\cdot,\cdot)$ denote distance in the graph Z. So $d(z_1, z_2)$ is the smallest number n such that there exists elements $s_1, .., s_n \in S$ with $s_1(...(s_n z_1)) = z_2$. Now,

$$\begin{aligned}
\mathbb{E}[\phi_j^2] &= \sum_{z_1,z_2\in Z} \mathbb{E}[\phi_{j,z_1}\phi_{j,z_2}] \\
&\le \sum_{z_1,z_2\in Z_n,\ d(z_1,z_2)>2n} \mathbb{E}[\phi_{j,z_1}\phi_{j,z_2}] + \sum_{z_1,z_2\in Z_n,\ d(z_1,z_2)\le 2n} \mathbb{E}[\phi_{j,z_1}\phi_{j,z_2}] \\
&\le \mu(C_j)^2|Z_n|^2 + |Z_n||S|^{2n}.
\end{aligned}$$

Thus,

$$Var(\phi_j) \leq |Z_n||S|^{2n} \leq |Z||S|^{2n}.$$

By Chebyshev's inequality,

$$Pr\Big[\ \Big|\frac{\phi_j(q)}{|Z|} - \frac{|Z_n|\mu(C_j)}{|Z|}\Big| > \epsilon_2/2\Big] < \frac{4|S|^{2n}}{|Z|\epsilon_2^2}.$$

If $r > n$ and $\delta \leq \epsilon_2/2$ then $\frac{|Z_n|}{|Z|} \geq 1 - \delta \geq 1 - \epsilon_2/2$. This implies

$$\Big|\frac{|Z_n|\mu(C_j)}{|Z|} - \mu(C_j)\Big| < \epsilon_2/2$$

for all $j : B(e,n) \to K$. Thus,

$$Pr\Big[\ \Big|\frac{\phi_j(q)}{|Z|} - \mu(C_j)\Big| > \epsilon_2\Big] < \frac{4|S|^{2n}}{|Z|\epsilon_2^2}.$$

If $|Z| > \frac{4|S|^{2n}}{\epsilon_2^3}$ then $\frac{4|S|^{2n}}{|Z|\epsilon_2^2} < \epsilon_2$. In this case,

$$Pr\Big[\ \Big|\frac{\phi_j(q)}{|Z|} - \mu(C_j)\Big| > \epsilon_2\Big] < \epsilon_2.$$

As mentioned above, if $|\frac{\phi_j(q)}{|Z|} - \mu(C_j)| \leq \epsilon_2$ for all $j : B(e,n) \to K$ then $\beta_q^{n-1} \in \mathcal{FA}(Z, \alpha^{n-1}, \epsilon)$. Thus

$$\begin{aligned} Pr[\beta_q^{n-1} \in \mathcal{FA}(Z,\alpha^{n-1},\epsilon)] &\geq 1 - \sum_j Pr\Big[\ \Big|\frac{\phi_j(q)}{|Z|} - \mu(C_j)\Big| > \epsilon_2\Big] \\ &\geq 1 - \epsilon_2 |K|^{|B(e,n)|} \geq 1 - \epsilon. \end{aligned}$$

In other words, if $Q_0 = \{q \in K^Z \mid \beta_q^{n-1} \in \mathcal{FA}(Z,\alpha^{n-1},\epsilon)\}$ then $\kappa^Z(Q_0) \geq 1-\epsilon$.

By the previous lemma, if $|Z|$ is sufficiently large then there exists a set $Q_1 \subset K^Z$ such that $\kappa^Z(Q_1) > 1 - \epsilon$ and for all $q \in Q_1$,

$$\exp(-[H(\kappa)+\epsilon]|Z|) \leq \kappa^Z(\{q\}) \leq \exp(-[H(\kappa)-\epsilon]|Z|).$$

Observe that $\kappa^Z(Q_0 \cap Q_1) \geq 1 - 2\epsilon$. So if ϵ, n are fixed and $|Z|, r$ are sufficiently large and δ is sufficiently small then

$$\begin{aligned} |\mathcal{FA}(Z,\alpha^{n-1},\epsilon)| &\geq |Q_0| \geq |Q_0 \cap Q_1| \geq \frac{\zeta(Q_0 \cap Q_1)}{\exp(-[H(\kappa)-\epsilon]|Z|)} \\ &\geq (1-2\epsilon)\exp([H(\kappa)-\epsilon]|Z|). \end{aligned}$$

This implies $F(\mathbf{Z}, \alpha^{n-1}, \epsilon) \geq H(\kappa) - \epsilon$. Since the right hand side does depend on n, $f(\mathbf{Z}, \alpha, \epsilon) \geq H(\kappa) - \epsilon$. Let $\epsilon \to 0$ to obtain $f(\mathbf{Z}, \alpha) \geq H(\kappa)$. □

The special case of theorem 1.1 in which G is finitely generated follows immediately from the above theorem and theorem 2.2. The general case is handled in section 9 below. This also proves corollary 1.2 when G is finitely generated except for the case in which one of K_1 or K_2 is infinite. That will be handled in the next section.

8 Products and the infinite entropy case

Proposition 8.1. *Let (X,μ) and (Y,ν) be two G-systems, each of which has a finite generating partition. Let $\mathbf{Z}$ be an approximating sequence for (G,S). If $(X,\mu)=(K^G,\kappa^K)$ is a Bernoulli-system with $|K|<\infty$ then*

$$f(\mathbf{Z}, X\times Y, \mu\times\nu) = f(\mathbf{Z},X,\mu) + f(\mathbf{Z},Y,\nu).$$

Proof. By corollary 6.2 and theorem 7.2, it follows that

$$f(\mathbf{Z}, X\times Y, \mu\times\nu) \le H(\kappa) + f(\mathbf{Z},Y,\nu) = f(\mathbf{Z},X,\mu) + f(\mathbf{Z},Y,\nu).$$

The proof of the opposite inequality is similar to the proof of the lower bound in theorem 7.2. Details are left to the reader. □

Proposition 8.2. *Let G be a finitely generated sofic Ornstein group. Let (K,κ) be a probability space with $H(\kappa)=+\infty$. Then either (K^G,κ^G) does not have a finite generating partition or, for every approximating sequence $\mathbf{Z}=\{(Z_i,\zeta_i)\}_{i=1}^\infty$ of (G,S), $f(\mathbf{Z},K^G,\kappa^G)=-\infty$.*

Proof. Let (K_2,κ_2) be a probability space with $0<H(\kappa_2)<\infty$. Consider the product probability space $(K\times K_2, \kappa\times\kappa_2)$. Observe that $H(\kappa\times\kappa_2)=+\infty$. Since G is an Ornstein group this implies that $((K\times K_2)^G, (\kappa\times\kappa_2)^G)$ is isomorphic to (K^G,κ^G). But $((K\times K_2)^G,(\kappa\times\kappa_2)^G)$ is isomorphic to the product space $(K^G,\kappa^G)\times(K_2^G,\kappa_2^G)$. Thus, by the previous proposition, if (K^G,κ^G) has a finite generating partition,

$$f(\mathbf{Z},K^G,\kappa^G) = f(\mathbf{Z},K^G,\kappa^G) + H(\kappa_2).$$

This implies $f(\mathbf{Z},K^G,\kappa^G)=-\infty$. □

9 Nonfinitely generated groups

Let G be a countable group. Let S be a symmetric, countable generating set for G. Let $S_1\subset S_2\subset \ldots.\subset S$ be an increasing sequence of finite symmetric subsets of S such that $\cup_n S_n = S$. Let G_n be the subgroup of G generated by S_n.

Let $\mathbf{Z}=\{Z_i\}_{i=1}^\infty$ be a sequence of S-graphs. Let (X,μ,α) be a G-process. For each n, let $f_n(\mathbf{Z},\alpha)$ be the f-value of the G_n-process (X,μ,α). I.e., $f_n(\mathbf{Z},\alpha)=f(\mathbf{Z},\alpha)$ where the acting group is G_n and each Z_i is interpreted as an S_n-graph in the obvious way. Define

$$f_\infty(\mathbf{Z},\alpha) := \liminf_{n\to\infty} f_n(\mathbf{Z},\alpha).$$

Note that $f_\infty(\mathbf{Z},\alpha)$ is not the same as $f(\mathbf{Z},\alpha)$.

Theorem 9.1. *If α and β are two generating partitions then $f_\infty(\mathbf{Z},\alpha)=f_\infty(\mathbf{Z},\beta)$. Hence we may define $f_\infty(\mathbf{Z},X,\mu)=f_\infty(\mathbf{Z},\alpha)$ for any generating partition α. This defines an isomorphism invariant.*

Lemma 9.2. *Let G be a countable group with symmetric generating set S. Let (X, μ) be a G-system. Let $\mathcal{P}_o$ denote the space of ordered partitions of X. Let $\phi : \mathcal{P}_o \to \mathbb{R}$ be a function that is invariant under splittings (i.e., $\phi(\alpha) = \phi(\sigma)$ whenever $\alpha \le \sigma \le \alpha \vee s\alpha$ for some $s \in S$) and is weakly upper semi-continuous. Then, for any two generating partitions α and β, $\phi(\alpha) = \phi(\beta)$.*

Proof. This was proven implicitly in [Bo08]. By [Bo08, proposition 4.3], any two topologically equivalent partitions have a common splitting. Hence if α and β are topologically equivalent, then $\phi(\alpha) = \phi(\beta)$. By the proof of [Bo08, theorem 3.4], if α and β are any finite generating partitions then there exists a sequence of partitions α_i, each topologically equivalent to α such that $\lim_{i\to\infty} d_o(\alpha_i, \beta) = 0$ and the number of atoms of α_i is bounded by $|\beta| + |\beta|^2$. So the weak upper semi-continuity of $\phi(\cdot)$ implies that, in general, $\phi(\alpha) \le \phi(\beta)$. Of course, the opposite inequality also holds from which the lemma follows. □

Proof of theorem 9.1. The theorem follows by verifying the hypotheses of the previous lemma for $\phi = f_\infty(\mathbf{Z}, \cdot)$.

First, if σ is any splitting of α then $f_\infty(\mathbf{Z}, \sigma) = f_\infty(\mathbf{Z}, \alpha)$. To see this, let $s \in S$ be such that $\alpha \le \sigma \le \alpha \vee s\alpha$. Then for all n such that $s \in S_n$, $f_n(\mathbf{Z}, \sigma) = f_n(\mathbf{Z}, \alpha)$. To see this, observe let $F_n(\mathbf{Z}, \alpha) = F(\mathbf{Z}, \alpha)$ where the later is defined with respect to G_n rather than G. Then F_n is monotone decreasing under splittings by proposition 4.1. But [Bo08, proposition 4.3], this implies that $f_n(\mathbf{Z}, \alpha)$ is the infimum of $F_n(\mathbf{Z}, \gamma)$ over all splittings γ of α. Of course, this implies the claim.

Second, $f_\infty(\mathbf{Z}, \cdot)$ is weakly upper semi-continuous on the space of partitions. This is immediate because each $f_n(\mathbf{Z}, \cdot)$ is weakly upper semi-continuous on the space of partitions and that a liminf of weakly upper semi-continuous functions is weakly upper semi-continuous.

□

Corollary 9.3. *Let G be a countable sofic group and let $(K_1, \kappa_1), (K_2, \kappa_2)$ be two probability spaces with K_1, K_2 both finite. If (K_1^G, κ_1^G) is isomorphic to (K_2^G, κ_2^G) then $H(\kappa_1) = H(\kappa_2)$.*

If G is Ornstein then the finiteness condition on K_1 and K_2 can be removed.

Proof. Let S be a countable symmetric generating set for G and let S_n be defined as above. Because G is sofic, for every n there exists an S_n-graph Z_n that is a $(n, 1/n)$-approximation to (G_n, S_n). For each $s \notin S_n$, choose an arbitrary bijection $s : \mathrm{Dom}(s) \to \mathrm{Rng}(s)$ with $\mathrm{Dom}(s), \mathrm{Rng}(s) \subset Z_n$. So each Z_n is now an S-graph. Let $\mathbf{Z} = \{Z_n\}_{n=1}^\infty$.

For $i = 1, 2$, let α_i be the canonical partition associated to K_i. So $\alpha_i = \{A_k \mid k \in K_i\}$ where $A_k = \{x \in K_i^G \mid x(e) = k\}$.

If K_1 and K_2 are finite then it follows from theorem 7.2 that $f_n(\mathbf{Z}, \alpha_i) = H(\kappa_i)$. Hence $f_\infty(\mathbf{Z}, \alpha_i) = H(\kappa_i)$. The previous theorem now implies the first part of the corollary.

For the second part, assume that G is Ornstein. If $H(\kappa_1) = H(\kappa_2)$ then the two processes (K_1^G, κ_1^G) and (K_2^G, κ_2^G) are isomorphic by the Ornstein property. If $H(\kappa_1)$ and $H(\kappa_2)$ are both finite then it follows from the Ornstein property that for $i = 1, 2$, (K_i^G, κ_i^G) is isomorphic to a Bernoulli shift of the form (L_i^G, λ_i^G) where L_i is finite. So the previous paragraph implies the result in this case.

Now suppose $H(\kappa_1) < H(\kappa_2) = +\infty$. If (K_2^G, κ_2^G) does not have a finite generating partition then we are done since (K_1^G, κ_1^G) does have a finite generating partition (because (K_1^G, κ_1^G) is isomorphic to (L^G, λ^G) for some probability space (L, λ) with L finite) and therefore they cannot be isomorphic. If (K_2^G, κ_2^G) has a finite generating partition α then by proposition 8.2, $f_n(\mathbf{Z}, \alpha) = -\infty$ for all n. Therefore, $f_\infty(\mathbf{Z}, K_2^G, \kappa_2^G) = f_\infty(\mathbf{Z}, \alpha) = -\infty$. Since $f_\infty(\mathbf{Z}, K_1^G, \kappa_2^G) \geq 0$, the previous theorem implies that (K_1^G, κ_1^G) and (K_2^G, κ_2^G) are not isomorphic. □

The corollary above immediately implies both theorem 1.1 and corollary 1.2.

Corollary 9.4. *Let G and $\mathbf{Z}$ be as above. Let (K^G, κ^G) be a Bernoulli shift over G with K finite. If (Y, ν) is a factor of (K^G, κ^G) then $f_\infty(\mathbf{Z}, Y, \nu) \geq 0$.*

Proof. Let α be the canonical partition of K^G: $\alpha = \{A_k \mid k \in K\}, A_k = \{x \in K^G \mid x(e) = k\}$. By theorem 7.2, $H(\alpha) = H(\kappa) = f_n(\mathbf{Z}, K^G, \kappa^G)$ for all n. So corollary 6.2 implies $f_n(\mathbf{Z}, Y, \nu) \geq 0$ for all $n \geq 0$. Hence $f_\infty(\mathbf{Z}, Y, \nu) \geq 0$. □

Proof of theorem 1.4. Let (L, λ) be a probability space with $0 < H(\lambda) < \infty$. By theorem 1.3, (L^G, λ^G) factors onto (K^G, κ^G). By the previous corollary, if $\mathbf{Z}$ is as above then either (K^G, κ^G) does not have a finite generating partition or $f_\infty(\mathbf{Z}, K^G, \kappa^G) \geq 0$ for any finite approximating sequence $\mathbf{Z}$. The result now follows from proposition 8.2. □

10 Conclusions

The definition of $f(\mathbf{Z}, \alpha)$ can be considerably generalized as follows. As in the rest of the paper, let G be a countable group acting by measure-preserving transformations on a probability space (X, μ). Let $\mathbf{C} = \{C_i\}_{i=1}^\infty$ where each $C_i = \{Z_{i,1}, ..., Z_{i,n_i}\}$ is a collection of finite S-graphs such that for every $i \geq 0$ and all $j = 1...n_i$, $|Z_{i,1}| = |Z_{i,j}|$. Given an ordered partition $\alpha = (A_1, ..., A_u)$ of X and $\epsilon > 0$, define

$$\mathcal{FA}(C_i, \alpha, \epsilon) = \sqcup_{j=1}^{n_i} \mathcal{FA}(Z_{i,j}, \alpha, \epsilon),$$

$$F(\mathbf{C}, \alpha, \epsilon) := \limsup_{i \to \infty} \frac{1}{|Z_{i,1}|} \ln \Big(\frac{|\mathcal{FA}(C_i, \alpha, \epsilon)|}{n_i} \Big),$$

$$F(\mathbf{C}, \alpha) := \lim_{\epsilon \to 0} F(\mathbf{C}, \alpha, \epsilon),$$

$$f(\mathbf{C}, \alpha) := \lim_{n \to \infty} F(\mathbf{C}, \alpha^n).$$

Theorem 2.2 (that $f(\mathbf{Z}, \cdot)$ is constant on generating partitions) holds for this more general definition and the proof is essentially the same. Indeed, all of the results and statements of this paper hold with $f(\mathbf{C}, \cdot)$ in place of $f(\mathbf{Z}, \cdot)$ with one exception: the value of $f(\mathbf{C}, \alpha)$ can be finite and negative. The proofs require only obvious minor changes.

In a forthcoming paper, I intend to show that the f-invariant defined in [Bo08] is of this form. To be precise, let $G = \langle s_1, ..., s_r \rangle$ be a free group. Let C_i denotes the collection of all

G-actions of the form $G \curvearrowright G/H$ where $H < G$ is a subgroup of index i. Let (X, μ) be an ergodic G-system with no nontrivial finite factors and let α be a finite partition of X. Then $f(\mathbf{C}, \alpha)$ is the same as $f(\alpha)$ as defined in [Bo08].

Because the f-invariant in [Bo08] can take on finite negative values this also shows that there exists a system (X, μ), a partition α and approximating sequences $\mathbf{Z}_1, \mathbf{Z}_2$ with $f(\mathbf{Z}_1, \alpha) \neq f(\mathbf{Z}_2, \alpha)$.

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