Sofic Groups and Bernoulli Shifts

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Cayley Graphs

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The **Cayley Graph** of (G, S) has vertex set G and $\forall g \in G, s \in S$, there is a directed edge from g to gs labeled s.

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Examples: $\mathbb{Z}^d$, solvable groups, Baumslag-Solitar groups.
Nonamenable groups: free groups, fundamental groups of closed hyperbolic manifolds, $SL_n(\mathbb{Z})$ for $n \geq 2$.

Residually Finite Groups

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All linear groups (i.e., subgroups of $GL_n(F)$ for any field F) are residually finite, (Mal'cev, 1940).

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Z is a **sofic** (r, δ)**-approximation to** (G, S) if $|Z_r| \geq (1 - \delta)|Z|$.

Sofic Groups (continued)

An **approximating sequence $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ for** (G, S) is such that each Z_i is an (r_i, δ_i)-approximation to (G, S) where $r_i \to \infty$ and $\delta_i \to 0$.

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G is **sofic** if there exists an approximating sequence for (G, S) (for some and hence every S). More generally, G is **sofic** if every finitely generated subgroup of G is sofic.

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- Let $H < G$ be normal and finite index. Let $Z = G/H$. If the quotient map $\pi : G \to G/H$ is $1-1$ on $B(e, r)$ then Z is a $(r, 0)$-approximation to (G, S).

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- Let $H < G$ be normal and finite index. Let $Z = G/H$. If the quotient map $\pi : G \to G/H$ is $1-1$ on $B(e,r)$ then Z is a $(r,0)$-approximation to (G,S).

Theorem (Elek and Szabo, 2006)

Residually amenable groups are sofic. But there exists a finitely generated sofic group that is not residually amenable. It is not known whether or not every group is sofic.

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- Two systems $(G_1, X_1, \mu_1), (G_2, X_2, \mu_2)$ are **isomorphic** if there exists a measure-space isomorphism $\phi : X_1 \to X_2$ and an isomorphism $h : G_1 \to G_2$ with $\phi(gx) = h(g)\phi(x)$ for a.e. $x \in X_1, g \in G$.

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- Main Problem: Classify systems up to isomorphism!

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- (G, K^G, κ^G) is the **Bernoulli shift over** G **with base measure** κ **.**

Bernoulli shifts over amenable groups

Definition

Let (K, κ) be a probability space. If there exists a countable or finite set $K' \subset K$ with $\kappa(K') = 1$ then define

$$H(\kappa) = -\sum_{k \in K'} \kappa(\{k\}) \log(\kappa(\{k\})).$$

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Theorem (Kolmogorov, Sinai, et al, 1958-)

If G is amenable and $(G, K_1^G, \kappa_1^G) \cong (G, K_2^G, \kappa_2^G)$ then $H(\kappa_1) = H(\kappa_2)$.

The converse

Definition

G is an **Ornstein group** if whenever $(K_1, \kappa_1), (K_2, \kappa_2)$ are probability spaces with $H(\kappa_1) = H(\kappa_2)$ then $(G, K_1^G, \kappa_1^G) \cong (G, K_2^G, \kappa_2^G)$.

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5. *(Open ?): Is every countable group Ornstein?*

New Results

Theorem

If G is sofic, $(G, K_1^G, \kappa_1^G) \cong (G, K_2^G, \kappa_2^G)$ and K_1, K_2 are finite then $H(\kappa_1) = H(\kappa_2)$. If G is also Ornstein then the finiteness condition can be removed.

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Theorem

If G contains a nonabelian free group then for every $(K_1, \kappa_1), (K_2, \kappa_2)$ with $H(\kappa_1)H(\kappa_2) > 0$, (G, K_1^G, κ_1^G) factors onto (G, K_2^G, κ_2^G) and vice versa (i.e., all Bernoulli shifts over G are **weakly isomorphic***).*

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- For $r \geq 0$, let Z_r be the set of all $z \in Z$ such that for any $s_1, .., s_r, t_1, .., t_r \in S \cup \{e\}$, $T^{s_1} ... T^{s_r} z$ is well-defined and

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- Z is an (r, δ)**-approximation to** (G, S) if $|Z| < \infty$ and $|Z_r| \geq (1 - \delta)|Z|$.

The f-invariants

Definition

Let (G, X, μ) be a system and $\alpha = (A_1, .., A_u)$ a finite ordered partition of X. Let Z be an S-graph and $\beta = (B_1, ..., B_u)$ an ordered partition of Z.
Then β **ϵ-approximates** α if

$$\sum_{s \in S} \sum_{i,j=1}^{u} \left| \mu(A_i \cap sA_j) - \zeta\big(B_i \cap T^s(B_j \cap \mathrm{Dom}(T^s))\big) \right| < \epsilon$$

where ζ is the uniform probability measure on Z.
Let $\mathcal{F}A(Z, \alpha, \epsilon)$ be the set of all ordered partitions β on Z that ϵ-approximate α.

The f-invariants (continued)

Definition

Let $\mathbf{Z} = \{Z_i\}_{i=1}^{\infty}$ be a sequence of S-graphs with $|Z_i| \to \infty$.

$$F(\mathbf{Z}, \alpha, \epsilon) := \limsup_{i \to \infty} \frac{1}{|Z_i|} \log |\mathcal{F}A(Z_i, \alpha, \epsilon)|,$$

$$F(\mathbf{Z}, \alpha) := \lim_{\epsilon \to 0} F(\mathbf{Z}, \alpha, \epsilon),$$

$$f(\mathbf{Z}, \alpha) = \lim_{n \to \infty} F(\mathbf{Z}, \alpha^n)$$

where

$$\alpha^n = \bigvee_{g \in B(e,n)} g\alpha.$$

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Theorem

- *If α, β are finite generating partitions then $f(\mathbf{Z}, \alpha) = f(\mathbf{Z}, \beta)$. Hence we can define $f(\mathbf{Z}, X, \mu) = f(\mathbf{Z}, \alpha)$ for any generating partition α.*
- *If $\mathbf{Z}$ is an approximating sequence (so G is sofic) then $f(\mathbf{Z}, K^G, \kappa^G) = H(\kappa)$ whenever K is finite.*

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Proposition

If α is generating then the set of all partitions β such that $\beta =_{top} \alpha$ is dense in the space of all generating partitions.

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Proposition

If $\alpha =_{top} \beta$ then for all n sufficiently large, α^n is a splitting of β (as well as of α).

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If σ is a splitting of α, then $F(\mathbf{Z}, \sigma) \leq F(\mathbf{Z}, \alpha)$.

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If α,β are generating partitions then $f(\mathbf{Z},\alpha) = f(\mathbf{Z},\beta)$.