MULTIPLE SOLUTIONS FOR A NONLINEAR DIRICHLET PROBLEM VIA MORSE INDEX

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In this lecture we study the nonlinear Dirichlet problem

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where Δ is the Laplacian operator, $\Omega \subset \mathbb{R}^N$ is a bounded region with smooth boundary, and $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 asymptotically linear, i.e.

$$f'(\infty) := \lim_{|t| \to \infty} f'(t) \in \mathbb{R}.$$

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As it is well known, to solve (0.1) is equivalent to find critical points of the functional $J : H_0^1(\Omega) \longrightarrow \mathbb{R}$ defined by

$$J(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - F(u) \right) dx, \tag{0.2}$$

where $F(\xi) = \int_0^\xi f(s)\, ds$.

Throughout this paper we assume f satisfies the following hypotheses:

(f1) $f(0) = 0$ and $f'(0) < \lambda_1$,

(f2) $f'(\infty) \in (\lambda_k, \lambda_{k+1})$,

(f3) There is a constant γ such that $f'(t) \leq \gamma < \lambda_{k+1}$ for every $t \in \mathbb{R}$.

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Under these conditions that we are assuming on f, $J \in C^2$.

We recall that if u is a critical point of J and there exists a maximal integer q such that $D^2J(u)$ is negative definite on some q-dimensional subspace then the **Morse index** $m(u, J)$ of u is q.

Theorem A. *If* f *satisfies (f1), (f2), (f3),* $k \geq 4$ *is an even number, and all the solutions of (0.1) are nondegenerate then problem (0.1) has at least five solutions:* $u_1 = 0$, $u_2 > 0$ *in* Ω, $u_3 < 0$ *in* Ω, u_4, *and* u_5.

Moreover,

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(c) If $m(u_5, J) = k$, there are two additional solutions u_6 and u_7 of (0.1). Moreover $m(u_6, J) = k - 1$.

(d) If u_4 is of one sign, there are also two additional solutions u_6 and u_7 of (0.1).

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We prove the existence of u_2 and u_3 using mountain pass arguments of the Ambrosetti-Rabinowitz type.

We show the existence of u_4 by using Lyapunov-Schmidt arguments to reduce the solvability of (0.1) to a finite dimensional problem.

We use degree theory to show the existence of u_5.

Sketch of the Proof of Theorem A:

Because of (f1), $u_1 \equiv 0$ is a strict local minimum of J, so

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The existence of the two signed-solutions $u_2 > 0$ in Ω and $u_3 < 0$ in Ω is proved by using the Mountain Pass Theorem. Moreover,

$$d_{loc}(\nabla J, u_2) = d_{loc}(\nabla J, u_3) = -1, \tag{0.3}$$

Sketch of the Proof of Theorem A:

Also, if S_2 (resp. S_3) is a region that contains all the positive (resp. negative) solutions of (0.1) and no one else, Castro and Cossio proved that

$$d(\nabla J, S_2, 0) = -1 = d(\nabla J, S_3, 0). \tag{0.4}$$

Since $f'(\infty)$ is not an eigenvalue of $-\Delta$, it can be shown that there exists an $R > 0$ such that $B_R(0)$ contains all the critical points of J, and

$$d(\nabla J, B_R(0), 0) = (-1)^k. \tag{0.5}$$

Sketch of the Proof of Theorem A:

Lemma 1. **(the Lyapunov-Schmidt reduction method)**

Let H be a real separable Hilbert space. Let X and Y be closed subspaces of H such that $H = X \oplus Y$. Let $J : H \to \mathbb{R}$ be a functional of class C^2. If there are $m > 0$ and $\alpha > 1$ such that

$$\langle \nabla J(x+y) - \nabla J(x+y_1), y - y_1 \rangle \geq m\|y - y_1\|^{\alpha} \quad \text{for all} \quad x \in X, y, y_1 \in Y \tag{0.6}$$

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then:

There exists a C^1 function $\psi : X \to Y$ such that

$$J(x + \psi(x)) = \min_{y \in Y} J(x+y).$$

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Moreover,

$$d_{loc}(\nabla J, x_0 + \psi(x_0)) = d_{loc}(\nabla \hat{J}, x_0). \tag{0.7}$$

$$m(u_0, J) = m(x_0, \hat{J}). \tag{0.8}$$

Sketch of the Proof of Theorem A:

Let X be the subspace of H spanned by $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ and let $Y := X^{\perp}$, thus $H = X \oplus Y$. Since $J \in C^2(H, \mathbb{R})$, there is a constant γ such that $f'(t) \leq \gamma < \lambda_{k+1}$ for every $t \in \mathbb{R}$, and

$$\langle \nabla J(u), v \rangle = \int_{\Omega} (\nabla u \cdot \nabla v - f(u)v)\, dx, \quad \forall u, v \in H_0^1(\Omega), \tag{0.9}$$

it follows that

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Because $f'(\infty) \in (\lambda_k, \lambda_{k+1})$ it follows that

$$\hat{J}(x) \longrightarrow -\infty \quad \text{as } \|x\| \longrightarrow \infty \quad \text{for } x \in X. \tag{0.11}$$

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Sketch of the Proof of Theorem A:

Observe that x_4 is a point of minimum of $-\hat{J}$. Thus, $d_{loc}(\nabla(-\hat{J}), x_4) = 1$ and $d_{loc}(\nabla \hat{J}, x_4) = (-1)^k$.

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In virtue of (0.7),

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Now, because of our hypothesis of nondegeneracy of critical points of J and the characterization of x_4, $m(x_4, \hat{J}) = k$. Lemma 1 implies that $m(u_4, J) = k$ and we have proved part (a) of Theorem A.

Sketch of the Proof of Theorem A:

Let K be the set of critical points of J. Let S_1, S_2, S_3 be disjoint open bounded regions in H such that $\overline{S_1} \cap K = \{u_1\}$, $\overline{S_2} \cap K$ is the set of positive solutions of (1), and $\overline{S_3} \cap K$ is the set of negative solutions of (0.1). We consider two cases, which will lead us to prove (b) and (d) of Theorem A.

Sketch of the Proof of Theorem A:

Suppose first that u_4 changes sign. Let S_4 be an open bounded region disjoint from $\overline{S_1 \cup S_2 \cup S_3}$ such that $\overline{S_4} \cap K = \{u_4\}$. Then, because of (0.4), (0.5), (0.12) and the excision property of Leray-Schauder degree,

$$\begin{aligned}
(-1)^k &= d(\nabla J, B_R(0), 0) \\
&= d(\nabla J, S_1, 0) + d(\nabla J, S_2, 0) + d(\nabla J, S_3, 0) + d(\nabla J, S_4, 0) \\
&\quad + d(\nabla J, B_R(0) \setminus \overline{S_1 \cup S_2 \cup S_3 \cup S_4}, 0) \\
&= 1 - 1 - 1 + 1 + d(\nabla J, B_R(0) \setminus \overline{S_1 \cup S_2 \cup S_3 \cup S_4}, 0).
\end{aligned}$$

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$$\begin{aligned}(-1)^k &= d(\nabla J, B_R(0), 0) \\ &= d(\nabla J, S_1, 0) + d(\nabla J, S_2, 0) + d(\nabla J, S_3, 0) + d(\nabla J, S_4, 0) \\ &\quad + d(\nabla J, B_R(0) \setminus \overline{S_1 \cup S_2 \cup S_3 \cup S_4}, 0) \\ &= 1 - 1 - 1 + 1 + d(\nabla J, B_R(0) \setminus \overline{S_1 \cup S_2 \cup S_3 \cup S_4}, 0).\end{aligned}$$

Therefore, $d(\nabla J, B_R(0) \setminus \overline{S_1 \cup S_2 \cup S_3 \cup S_4}, 0) = 1$.

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Since

$$\langle D^2J(u)y,y\rangle \geq (1-\frac{\gamma}{\lambda_{k+1}})\|y\|^2; \qquad \forall u \in H \quad \forall y \in Y. \tag{0.13}$$

it follows that $m(u_5,J) \leq k$. Observe that u_5 is a sign-changing solution of (0.1). This prove Theorem A part (b).

Sketch of the Proof of Theorem A:

Now suppose u_4 is of one sign. Without loss of generality we can assume $u_4 \in S_2$. Let P be a subregion of S_2 such that $\overline{P} \cap K = \{u_2, u_4\}$. From (0.3), (0.4), (0.12) and the excision property of Leray-Schauder degree it follows that

$$
\begin{aligned}
-1 &= d(\nabla J, S_2, 0) \\
&= d_{loc}(\nabla J, u_2) + d_{loc}(\nabla J, u_4) + d(\nabla J, S_2 \setminus \overline{P}, 0) \\
&= -1 + (-1)^k + d(\nabla J, S_2 \setminus \overline{P}, 0).
\end{aligned}
$$

Hence, the existence property of the Leray-Schauder degree implies there must be another signed solution of (0.1).

Sketch of the Proof of Theorem A:

Let us denote by u_5 this solution. Arguing as in the case in which u_4 changes sign, one can get the existence of u_6 and u_7 which are sign-changing solutions of (0.1). We have completed the proof of (b) and (d) in Theorem A.

Sketch of the Proof of Theorem A:

We finally prove (c). Let us suppose $m(u_5, J) = k$. Writing u_5 as $u_5 = x_5 + \psi(x_5)$, Lemma 1 asserts that $m(x_5, \hat{J}) = k$ and so x_5 is a local maximum of $\hat{J}$.

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Thus, x_4 and x_5 are points of minima of $-\hat{J}$. The Mountain Pass Theorem implies that there exists a critical point x_6 of the mountain pass type for $-\hat{J}$.

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Thus, x_4 and x_5 are points of minima of $-\hat{J}$. The Mountain Pass Theorem implies that there exists a critical point x_6 of the mountain pass type for $-\hat{J}$.

Because of nondegeneracy, $m(x_6, -\hat{J}) = 1$ (see [Hofer]). Hence, $m(x_6, \hat{J}) = k - 1$. Again, by Lemma 1 $u_6 = x_6 + \psi(x_6)$ is a critical point of J whose Morse index is $k - 1$. Finally, using a degree counting, as before, we get u_7. In this way we got (c), and our proof is complete. ▲

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