Linear Differential Operators and the Distribution of Zeros of Polynomials

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Abstract. The purpose of this paper is fourfold: (1) to survey some classical and recent results in the theory of distribution of zeros of entire functions, (2) to demonstrate a novel proof answering a question of Raitchinov, (3) to present some new results in the theory of complex zero decreasing operators, and (4) to initiate the study of the location of zeros of complex polynomials under the action of certain linear operators. In addition, several open problems are given.

0. Introduction

This paper is organized under the following section headings:

1991 *Mathematics Subject Classification.* Primary 26C10, 30C15, 30C10; Secondary 26D10.

Key words and phrases. Linear Differential Operators, The Laguerre-Pólya class, zeros of entire functions.

The principal theme of our present work centers around the investigation of the distribution of zeros of complex polynomials and certain classes of transcendental real entire functions under the action of linear operators. In order to make this paper self-contained, we commence Section 1 with a review of the background material that will be needed in the sequel. In Section 1.1, we introduce some definitions, nomenclature and state several outstanding open problems (see, for example, Problems 1.3–1.11, Problems 1.18–1.19, Problem 1.25 and Problems 1.30–1.33). The remarkable fact that any linear operator acting on the vector space of complex polynomials can be represented as a formal series of linear differential operators with complex polynomial coefficients is established in Section 1.2 (see Theorem 1.14). Since the class of entire functions, known as the *Laguerre-Pólya class* plays a pivotal role in our investigation, we recall in Section 1.3 (and establish in subsequent sections) several facts about this class of functions. We associate with functions in the Laguerre-Pólya class two important families of linear operators; these are termed in the literature, multiplier sequences and complex zero decreasing sequences (Section 1.4). The main goal of Section 2 is to establish results that extend the action of linear operators from the vector space of polynomials to some classes of transcendental entire functions. In Section 3, we prove generalizations of the Hermite-Poulain Theorem. In addition, we investigate some extensions of Laguerre's Theorem in Section 4. Section 5 is the highlight of this paper, wherein we study the determination of linear transformations by means of the characterization of their associated complex polynomials. We answer a question of Raitchinov with a novel proof, that is a refinement of a result by Djokovic, and establish new results in the theory of complex decreasing operators. The aforementioned results focus on the *number* of non-real zeros of real polynomials and a specific class of transcendental entire functions. The purpose of Section 6 and Section 7, is to complement the above work and initiate the study of the *location* of zeros of complex polynomials under the action of certain linear operators. In particular, we also consider some infinite order differential operators which preserve positivity (cf. Example 6.10, Definition 6.12 and Corollary 6.13). The material covered in Sections 1–7 is supplemented by an appendix (Section 8).

1. Background Information

In this section we introduce some definitions and nomenclature to be used in the sequel (Section 1.1). We also list several outstanding open problems relevant to our investigations (Problems 1.3–1.11, Problems 1.18–1.19, Problem 1.25 and Problems 1.30–1.33). We establish some results pertaining to the characterization and representation of linear operators acting on the vector space of polynomials (Section 1.2). An important class of transcendental entire functions which plays a fundamental role in our investigations is the Laguerre-Pólya class (cf. Definition 1.21). In addition, in this section (Section 1.3), we highlight a number of the fundamental properties of functions in the Laguerre-Pólya class. In Section 1.4, we describe the significant classes of linear operators known as multiplier sequences and complex zero decreasing sequences.

1.1. **Definitions and Open Problems.**

Let $D = d/dz$ denote differentiation with respect to z. In general, if

$$h(y) = \sum_{k=0}^{\infty} Q_k(z) y^k \quad (Q_k(z) \in \mathbb{C}[z]; k = 0, 1, 2, \ldots)$$

is a formal power series, then we define the action of the linear operator, $h(D)$, on an entire function $f(z)$ by

$$h(D)[f(z)] := \sum_{k=0}^{\infty} Q_k(z) f^{(k)}(z), \tag{1.1}$$

whenever the right hand side of (1.1) represents an analytic function in some neighborhood of the origin. In general, $h(D)[f(z)]$ need not represent an entire function. For example, if $h(y) = e^{-y^2}$ and $f_\alpha(z) = e^{-\alpha z^2}$, then, for $0 < \alpha < 1/4$, we can show that

$$h(D)[f_\alpha(z)] = \left(\sum_{k=0}^{\infty} \binom{2k}{k} \alpha^k \right) \exp\left(-\frac{\alpha z^2}{1 - 4\alpha} \right). \tag{1.1a}$$

In the Appendix (cf. Proposition 8.1), we give two proofs of the equation (1.1a); one of the proofs appears to be new. We remark that it follows from [46, Lemma 3.1] that, if $0 < \alpha < 1/4$, then $h(D)[f_\alpha(z)]$ is an entire function. However, no explicit formula is given in [46].

Definition 1.1. A non-zero univariate polynomial with real coefficients is called *hyperbolic* (or is in the Laguerre-Pólya class, see Definition 1.21) if all its zeros are real (see [21], [23], [42], [46] and [123]). A univariate polynomial f with complex coefficients is called *stable* if $|f(z)| > 0$ whenever $\mathrm{Im}(z) > 0$. Hence, a univariate polynomial with real coefficients is stable if and only if it is hyperbolic.

Notation 1.2. Let $\Omega \subset \mathbb{C}$, and denote by $\pi(\Omega)$ the class of all (complex or real) univariate polynomials whose zeros lie in Ω. Thus, if $f(x) \in \pi(\mathbb{R})$, then $f(x)$ is a hyperbolic polynomial; that is, $f(x)$ has only real zeros. Let π_n denote the vector space (over $\mathbb{R}$ or $\mathbb{C}$) of all polynomials of degree $\leq n$, and let $\pi_n(\Omega)$ denote the class of all polynomials of degree $\leq n$, all of whose zeros lie in Ω.

In 2007, at the American Institute of Mathematics Workshop entitled *Pólya-Schur-Lax Problems: Hyperbolicity and Stability Preservers* ([5]), there were over 48 problems proposed by the organizers and the participating researchers. Here we will confine our attention to a select few problems as they are related to our investigation (see also [50]).

Problem 1.3 (cf. Notation 1.2)**.** Characterize all linear operators (transformations)

$$T : \pi(\Omega) \to \pi(\Omega) \cup \{0\}. \tag{1.2}$$

From a historical perspective, it is interesting to note that finding just one new T satisfying (1.2) can be significant. For example, if Ω is a convex region in $\mathbb{C}$ and $T = D$, where $D = \frac{d}{dz}$, then by the classical Gauss-Lucas Theorem (Theorem 7.2), T satisfies (1.2) (cf. [109, p. 22], [38] and [111]). In the sequel, as we consider some special cases of Problem 1.3, we will encounter some other notable linear transformations which satisfy (1.2). The finite analog of Problem 1.3 may be formulated as follows.

Problem 1.4 (cf. Notation 1.2)**.** Describe all linear operators

$$T : \pi_n(\Omega) \to \pi(\Omega) \cup \{0\} \quad \text{for } n \in \mathbb{N}. \tag{1.3}$$

We remark that Problems 1.3 and 1.4 originate from the works of Laguerre and Pólya-Schur [123]. See also J. Borcea, P. Brändén, B. Shapiro [23] and Q. I. Rahman, G. Schmeisser [127, pp. 182-183]. Various special cases of Problem 1.3 when $\Omega = \mathbb{R}$ have been considered, for example, by A. Aleman, D. Beliaev, H. Hedenmalm [3] and S. Fisk [64]. For additional information, see [21]-[24], [30], [42], [46] and the references contained therein.

Definition 1.5. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is said to be a *hyperbolicity preserver* if, whenever $f(x) \in \mathbb{R}[x]$ is hyperbolic, then the polynomial $T[f(x)]$ is also hyperbolic; that is, $T : \pi(\mathbb{R}) \to \pi(\mathbb{R})$.

Problem 1.6. Characterize all linear transformations $T : \pi(\mathbb{R}) \to \pi(\mathbb{R})$.

Problem 1.7. Characterize all linear transformations $T : \pi_n(\mathbb{R}) \to \pi_n(\mathbb{R})$.

In [23], J. Borcea, P. Brändén and B. Shapiro completely solved Problems 1.3 and 1.4 for all closed circular domains and their boundaries. In [22], they obtained multivariate extensions for all finite order linear differential operators with polynomial coefficients. The following cases remain open.

Problem 1.8. Settle Problems 1.3 and 1.4 in the important special cases (i) Ω is an open circular domain, (ii) Ω is a sector or a double sector, (iii) Ω is a strip, and (iv) Ω is a half-line.

Definition 1.9. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is called a *complex zero decreasing operator* if, for any real polynomial $f(x)$,

$$Z_c(T[f(x)]) \le Z_c(f(x)),$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

Problem 1.10. Characterize all complex zero decreasing operators.

Problem 1.11 ([46])**.** Let π_n denote the vector space over $\mathbb{R}$ of all real polynomials of degree $\le n$. Characterize all linear transformations $T : \pi_n \to \pi_n$ such that

$$Z_c(T[f(x)]) \le Z_c(f(x)) \tag{1.4}$$

for all $f(x) \in \pi_n$.

In light of Definitions 1.5 and 1.9, any complex zero decreasing operator is a hyperbolicity preserver. The quintessential example of a complex zero decreasing operator is the differential operator, $D = d/dx$, which satisfies (1.4) as a consequence of Rolle's Theorem. Another example of a linear operator that satisfies (1.4) is given by the following classical result.

Theorem 1.12 (Hermite-Poulain, [111])**.** *Let $h(y) = \sum_{k=0}^{n} c_k y^k$ be a real polynomial with only real zeros. Then the linear operator $h(D) : \mathbb{R}[x] \to \mathbb{R}[x]$ (see (1.1)) is a complex zero decreasing operator.*

Proof. First, we consider the case when $h(y)$ is of degree one; that is, $h(y) = y + \alpha$ for some $\alpha \in \mathbb{R}$. Let $f(x) \in \mathbb{R}[x]$ be a polynomial with m real and $2M$ non-real zeros. Since $e^{\alpha x}$ is an entire function with no (real) zeros, $e^{\alpha x} f(x)$ must have

m real zeros, the same number as $f(x)$. It follows from Rolle's Theorem that $(e^{\alpha x}f(x))' = e^{\alpha x}(\alpha f(x) + f'(x))$ has at least $m-1$ real zeros. Consequently, $\alpha f(x) + f'(x)$ has at least $m-1$ real zeros, and since the degree of $\alpha f(x) + f'(x)$ is not greater than $m + 2M$,

$$Z_c(\alpha f(x) + f'(x)) \leq Z_c(f(x)) + 1 = 2M + 1. \tag{1.5}$$

But the polynomial $\alpha f(x)+f'(x)$ cannot have exactly $2M+1$ non-real zeros since the number of non-real zeros of a real polynomial must be an even number. Therefore, $Z_c((\alpha + D)f(x)) \leq Z_c(f(x))$. Now, let $h(y) = \sum_{k=0}^{n} c_k y^k = c\prod_{k=1}^{n}(y + \alpha_k)$ and observe the following relations.

$$\begin{aligned}
f_1(x) &= (\alpha_1 + D)[f(x)] = \alpha_1 f(x) + f'(x),\\
f_2(x) &= (\alpha_2 + D)[f_1(x)] = \alpha_2\alpha_1 f(x) + (\alpha_2 + \alpha_1)f'(x) + f''(x),\\
\ldots f_n(x) &= (\alpha_n + D)[f_{n-1}(x)] = \alpha_n f_{n-1}(x) + f'_{n-1}(x)\\
&= \alpha_1 \cdots \alpha_n f(x) + (\alpha_1 \cdots \alpha_{n-1} + \ldots)f'(x) + f^{(n)}(x).
\end{aligned}$$

Hence, $cf_n(x) = h(D)[f(x)]$, and by (1.5) we see that

$$Z_c(f) \geq Z_c(f_1) \geq \ldots \geq Z_c(f_n) = Z_c(h(D)[f]),$$

as desired. □

Example 1.13. In Theorem 1.12, the assumption that $h(y)$ is hyperbolic is necessary, as the following example shows. Let $h(y) = y^2 + 1$, having non-real zeros. The polynomial $f(x) = x^2 - 1$ has two real zeros and no non-real zeros. However, the polynomial $h(D)[f(x)] = f(x) + f''(x) = x^2 + 1$ has two non-real zeros.

1.2. **Representation of Linear Operators.** It is quite remarkable that *any* linear operator, $T : \mathbb{C}[z] \to \mathbb{C}[z]$, can be represented as a formal series differential operator with complex polynomial coefficients. These polynomials are defined recursively, as the proof of the following theorem demonstrates.

Theorem 1.14 ([116, p. 32])**.** *Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator. Then there exists a unique sequence of complex polynomials, $\{Q_k(z)\}_{k=0}^{\infty}$, such that*

$$T[f(z)] = \sum_{k=0}^{\infty} Q_k(z) f^{(k)}(z) \tag{1.6}$$

for all $f(z) \in \mathbb{C}[z]$.

Proof. Let T be a linear operator on the set of complex polynomials. Define the sequence $\{Q_k(z)\}_{k=0}^{\infty}$ recursively by:

$$Q_0(z) = T[1],$$

and for $n \geq 1$,

$$Q_n(z) = \frac{1}{n!}\left(T[z^n] - \sum_{k=0}^{n-1} Q_k(z) D^k z^n\right). \tag{1.7}$$

Note that $n!Q_n(z) = Q_n(z)D^n z^n$. Thus, $T[z^n] = n!Q_n(z) + \sum_{k=0}^{n-1} Q_k(z)D^k z^n = \sum_{k=0}^{n} Q_k(z)D^k z^n$. Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a complex polynomial. By the linearity of T,

$$T[f(z)] = T\left[\sum_{k=0}^{n} a_k z^k\right] = \sum_{k=0}^{n} a_k T[z^k].$$

Therefore,

$$T[f(z)] = \sum_{k=0}^{n} a_k \left(\sum_{j=0}^{k} Q_j(z)D^j[z^k]\right) = \sum_{k=0}^{n} \left(\sum_{j=0}^{k} a_k Q_j(z)D^j[z^k]\right).$$

Since $D^j[z^k] = 0$ when $k > j$,

$$T[f(z)] = \sum_{k=0}^{n} \left(\sum_{j=0}^{n} a_k Q_j(z)D^j[z^k]\right).$$

Interchanging the order of summation yields

$$\begin{aligned} T[f(z)] &= \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_k Q_j(z)D^j[z^k]\right) = \sum_{j=0}^{n} Q_j(z) \left(\sum_{k=0}^{n} a_k D^j[z^k]\right) \\ &= \sum_{j=0}^{n} Q_j(z)D^j \left[\sum_{k=0}^{n} a_k z^k\right] = \sum_{j=0}^{n} Q_j(z) f^{(j)}(z), \end{aligned}$$

as desired.

To show uniqueness, let $\{P_k(z)\}_{k=0}^{\infty}$ be a set of complex polynomials such that

$$T[f(z)] = \sum_{k=0}^{\infty} P_k(z) f^{(k)}(z)$$

for all $f \in \mathbb{C}[z]$. Then,

$$P_0(z) = T[1] = Q_0(z).$$

Suppose that for all $m \leq n$, $Q_m(z) = P_m(z)$. But note that when $f(z) = z^{n+1}$,

$$\sum_{k=0}^{n+1} Q_k(z)D^k[z^{n+1}] = T[z^{n+1}] = \sum_{k=0}^{n+1} P_k(z)D^k[z^{n+1}].$$

Since $Q_m(z) = P_m(z)$ for all $m \leq n$, $P_{n+1}(z)D^{n+1}[z^{n+1}] = Q_{n+1}(z)D^{n+1}[z^{n+1}]$, and consequently, $P_{n+1}(z) = Q_{n+1}(z)$. Hence, by induction, $P_k(z) = Q_k(z)$ for all $k \in \mathbb{N}$. $\square$

Remark 1.15. From the recursive formula (1.7), it follows that if a linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ acts on the space of real polynomials, then it can be represented as

$$T = \sum_{k=0}^{\infty} Q_k(x)D^k,$$

where $Q_k(x) \in \mathbb{R}[x]$.

To illustrate Theorem 1.14, we explicitly calculate the polynomials $Q_k(x)$ for some given linear operators.

Example 1.16. Let T be a linear operator defined by $T[x^n] = (5+3n+3n^2)x^n$. We now calculate the associated polynomials $Q_k(x)$.

$$\begin{aligned}
Q_0(x) &= T[1] = 5\\
Q_1(x)D[x] &= T[x] - Q_0(x)x = 11x - 5x = 6x\\
Q_2(x)D^2[x^2] &= T[x^2] - Q_1(x)D[x^2] - Q_0(x)x^2 = 23x^2 - 12x^2 - 5x^2 = 6x^2\\
Q_3(x)D^3[x^3] &= T[x^3] - Q_2(x)D^2[x^3] - Q_1(x)D[x^3] - Q_0(x)x^3\\
&= 41x^3 - 18x^3 - 18x^3 - 5x^3 = 0\\
Q_n(x)D^n[x^n] &= T[x^n] - \sum_{k=0}^{n-1} Q_k(x)D^k[x^n]\\
&= (5+3n+3n^2)x^n - Q_2(x)D^2[x^n] - Q_1(x)D[x^n] - Q_0(x)x^n\\
&= (5+3n+3n^2)x^n - (3n^3-3n)x^n - (6n)x^n - 5x^n\\
&= 0 \quad (n = 3, 4, \ldots).
\end{aligned}$$

We observe that $(5+6xD+3x^2D^2)[x^n] = 5x^n+6nx^n(3(n^2-n))x^n = (5+3n+3n^2)x^n$, and by the linearity of the differential operator, $T[f(x)] = (5+6xD+3x^2D^2)[f(x)] = \sum_{k=0}^{\infty} Q_k(x)f^{(k)}(x)$.

Example 1.17. Let $T : \mathbb{R}[x] \to \mathbb{R}$ be the linear operator (functional) defined by $T[f(x)] = \int_0^1 f(x)dx$. Then $Q_0(x) = T[1] = 1$, and

$$Q_1(x)Dx + Q_0(x)x = Q_1(x) + x = T[x] = \frac{1}{2},$$

thus $Q_1(x) = \frac{1}{2} - x$. Now, we next show that for $k \in \mathbb{N}$,

$$Q_k(x) = \sum_{i=0}^{k} \frac{(-1)^i x^i}{(k+1-i!)(i!)}.$$

A calculation shows that

$$\begin{aligned}
T[x^n] &= \sum_{k=0}^{n} (\sum_{i=0}^{k} \frac{(-1)^i x^i}{(k+1-i)!(i!)})D^k(x^n)\\
&= \sum_{k=0}^{n} \left(\sum_{i=0}^{k} \frac{(-1)^i x^{i+n-k}(n!)}{(k+1-i)!(i!)(n-k)!} \right)\\
&= \sum_{j=0}^{n} \frac{n!}{(n-j+1)!} \left(\sum_{i=0}^{j} \frac{(-1)^i}{i!(j-i)!} \right) x^j\\
&= \frac{n!}{(n+1)!} = \frac{1}{n+1},
\end{aligned}$$

which is the value of $\int_0^1 x^n dx$.

We will explore this characterization further in the subsequent sections; in particular, its partial extension to the Laguerre-Pólya class in Section 2 and the characterization of some classes of linear operators by way of the coefficients $Q_k(x)$ in Section 5.

Problem 1.18. Characterize the polynomials Q_k such that, if $T = \sum_{k=0}^{\infty} Q_k(z)D^k$, then

$$T : \pi_n(\Omega) \to \pi(\Omega) \cup \{0\} \quad \text{for } n \in \mathbb{N}.$$

Problem 1.19. Characterize the polynomials Q_k such that, if $T = \sum_{k=0}^{\infty} Q_k(z)D^k$, then T is a complex zero decreasing operator (cf. Definition 1.9).

Remark 1.20 ([116, p. 37])**.** We can make use of Theorem 1.14 to give a matrix representation of any linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$. Letting $f(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial, we see that if we represent $f(z)$ as the matrix

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ 0 \\ \vdots \end{pmatrix},$$

then we can represent the differential operator, $D = d/dz$, as the matrix

$$M_D = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Indeed,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & \dots \\ 0 & 0 & 0 & 3 & \dots \\ 0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1 & 2a_2 & \dots & na_n & 0 & \dots \end{pmatrix}.$$

Now, if we let

$$Q_k(z) = \sum_{i=0}^{\infty} c_{k,i} z^i \quad (k = 0, 1, 2, \dots),$$

be a sequence of polynomials (where $c_{k,i}$ is zero for all but finitely many i), then we may represent the linear operator $T = \sum_{k=0}^{\infty} Q_k(z)D^k$ by the following matrix:

$$M_T = \begin{pmatrix} c_{0,0} & c_{1,0} & 2!c_{2,0} & 3!c_{3,0} \dots & \\ c_{0,1} & c_{0,0}+c_{1,1} & 2c_{1,0}+2!c_{2,1} & (3\cdot 2)c_{2,0}+3!c_{3,1} & \dots \\ c_{0,2} & c_{0,1}+c_{1,2} & c_{0,0}+2c_{1,1}+2!c_{2,2} & 3c_{1,0}+(3\cdot 2)c_{2,1}+3!c_{3,2} & \dots \\ c_{0,3} & c_{0,2}+c_{1,3} & c_{0,1}+2c_{1,2}+2!c_{2,3} & c_{0,0}+3c_{1,1}+(3\cdot 2)c_{2,2}+3!c_{3,3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

1.3. **The Laguerre-Pólya Class.** Real entire functions which are the uniform limits on compact subsets of $\mathbb{C}$ of polynomials having all their zeros in some prescribed region have been studied by many authors (see [46] and the references contained therein). In particular, if each of the approximating polynomials has only real zeros, then the limit entire function must be of a very specific form.

Definition 1.21 ([42])**.** A real entire function $f(x) := \sum_{k=0}^{\infty} \gamma_k x^k/k!$ is said to be in the *Laguerre-Pólya class*, written $f(x) \in \mathcal{L}-\mathcal{P}$, if it can be expressed in the form

$$f(x) = cx^n e^{-\alpha x^2+\beta x} \prod_{k=1}^{\infty} \left(1+\frac{x}{x_k}\right) e^{-\frac{x}{x_k}},$$

where $c, \beta, \in \mathbb{R}$, $x_k \in \mathbb{R} \cup \{\infty\}$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} 1/x_k^2 < \infty$.

If $f(x) \in \mathcal{L}-\mathcal{P}$ has all its zeros in an interval (a, b) (or $[a, b]$), then we write that $f(x) \in \mathcal{L}-\mathcal{P}(a, b)$ (or $f(x) \in \mathcal{L}-\mathcal{P}[a, b]$). If $f(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{L}-\mathcal{P}$ and $\gamma_k \geq 0$ (or $(-1)^k\gamma_k \geq 0$ or $-\gamma_k \geq 0$) for all k, then it is said that $f(x) \in \mathcal{L}-\mathcal{P}$ is of type I in the Laguerre-Pólya class, and we write that $f \in \mathcal{L}-\mathcal{P}I$. If $f(x) = \sum_{k=0}^{\infty} \gamma_k x^k/k! \in \mathcal{L}-\mathcal{P}$ and $\gamma_k \geq 0$ for all k, then we write that $f(x) \in \mathcal{L}-\mathcal{P}^+$. If an entire function $f(x)$ can be written as the product $f(x) = p(x)\varphi(x)$, where $\varphi(x) \in \mathcal{L}-\mathcal{P}$ and $p(x)$ is a real polynomial, then we write that $f(x) \in \mathcal{L}-\mathcal{P}^*$.

Remark 1.22. The significance of this class of functions is that it satisfies many remarkable properties. For these we refer to [46] and the many references contained therein. We highlight a few of those properties that will be used in the sequel. Let

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \tag{1.8}$$

be a real entire function.

(i) The functions of the Laguerre-Pólya class, and *only these*, can be approximated uniformly on compact subsets of $\mathbb{C}$ by a sequence of polynomials with only real zeros (see Theorem 2.2, Theorem 2.3 and [103, Chapter VIII]).

(ii) It follows from (i) that the class $\mathcal{L}-\mathcal{P}$ is closed under differentiation; that is, if $\varphi(x) \in \mathcal{L}-\mathcal{P}$, then $\varphi^{(n)}(x) \in \mathcal{L}-\mathcal{P}$ for $n \geq 1$ (see Corollary 2.4).

(iii) If $\varphi \in \mathcal{L}-\mathcal{P}$, then the *Jensen polynomials*, $g_n(x)$, associated with $\varphi(x)$,

$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k, \quad n = 0, 1, 2, \dots,$$

have only real zeros (cf. [123]).

(iv) If $\varphi \in \mathcal{L}-\mathcal{P}$, then the *Laguerre inequality* holds; that is,

$$L_1[\varphi^{(p)}](x) = \varphi^{(p)}(x)^2 - \varphi^{(p-1)}(x)\varphi(p+1)(x) \geq 0 \tag{1.9}$$

for all $x \in \mathbb{R}$ and $p = 1, 2, \ldots$. Consequently, the derivative of the *logarithmic derivative* of φ, $\frac{\varphi'(x)}{\varphi(x)}$, is always negative, since

$$\left(\frac{\varphi'(x)}{\varphi(x)}\right)' = \frac{\varphi(x)\varphi''(x) - (\varphi'(x))^2}{(\varphi(x))^2} = \frac{-L_1[\varphi](x)}{(\varphi(x))^2}$$

must always be negative in light of (1.9).

(v) If $\varphi \in \mathcal{L}-\mathcal{P}$, then it follows from (iv) that the *Turán inequalities* hold; that is,

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0, \quad k = 1, 2, \ldots.$$

Moreover, two consecutive terms of the sequence $\{\gamma_j\}_{j=0}^{\infty}$ cannot be zero unless all the subsequent or preceding terms are zero; that is, if $\gamma_k^2 + \gamma_{k+1}^2 = 0$, then $\gamma_j = 0$ for all $j \geq k$ or $\gamma_j = 0$ for all $j \leq k+1$. Furthermore, if $\gamma_k = 0$, but $\gamma_{k-1}\gamma_{k+1} \neq 0$, then $\gamma_k\gamma_{k+1} < 0$ (see also [40], [55] and [56]).

Remark 1.23. For technical reasons, it is convenient to include the zero function $f(x) \equiv 0$ in the Laguerre-Pólya class, as it readily satisfies properties (i), (ii), (iv) and (v) of Remark 1.22. However, the zero function has non-real zeros, as it has a zero at every point on the complex plane.

Definition 1.24 ([46, Definition 3.10])**.** We define the *extended Laguerre expressions* in the following manner. For any real entire function $\varphi(x)$ and $k \geq 1$, set

$$\mathcal{T}_k^{(1)}(\varphi(x)) := (\varphi^{(k)}(x))^2 - \varphi^{(k-1)}(x)\varphi^{(k+1)}(x),$$

and for $n \geq k$, set

$$\mathcal{T}_k^{(n)}(\varphi(x)) := (\mathcal{T}_k^{(n-1)}(\varphi(x))^2 - \mathcal{T}_{k-1}^{(n-1)}(\varphi(x))\mathcal{T}_{k+1}^{(n-1)}(\varphi(x)).$$

Problem 1.25 (cf. [45] and [46, Problem 3.11])**.** If $\varphi(x) \in \mathcal{L}-\mathcal{P}^+$ (see Definition 1.21), are the *iterated Laguerre inequalities* valid for all $x \geq 0$? That is, is it true that the inequality

$$\mathcal{T}_k^{(n)}(\varphi(x)) \geq 0$$

holds for all $x \geq 0$ and all $k \geq n$?

1.4. **Multiplier Sequences and CZDS.** If the linear operator T is given by $T[x^k] = \gamma_k x^k$, for some real sequence $\{\gamma_k\}_{k=0}^{\infty}$, then the matrix representation of T (cf. Remark 1.20) must be a diagonal matrix with γ_k on the diagonal of the k^{th} row. For this reason, linear operators arising in this way are sometimes referred to as diagonal operators (see [35]). In particular, the study of hyperbolicity preservers (see Definition 1.5) of this form (see Definition 1.26) has been investigated by many authors (see [42], [46], [123] and [122]).

Definition 1.26. A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence* if, whenever the real polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ is hyperbolic, the polynomial $T[f(x)] = \sum_{k=0}^{n} \gamma_k a_k x^k$ is also hyperbolic.

The following result of Pólya and Schur completely characterizes multiplier sequences.

Theorem 1.27 (Pólya-Schur [123], [103, Chapter VIII] and [111, Kapitel II])**.** *A sequence $\{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence if and only if*

1. *(Transcendental Characterization)*

$$\varphi(x) = T[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L} - \mathcal{P}\mathrm{I}. \tag{1.10}$$

2. *(Algebraic Characterization)*

$$g_n(x) = T[(1+x)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \in \mathcal{L} - \mathcal{P}\mathrm{I} \textit{ for all } n = 1, 2, 3, \ldots. \tag{1.11}$$

The polynomials $g_n(x)$ are called the *Jensen polynomials* associated with the entire function $\varphi(x)$ in (1.10) (cf. Remark 1.22), and will be discussed in greater detail in Section 2. Although up to this point, multiplier sequences have only been defined on vector spaces of polynomials, it is possible to apply them to the *transcendental* entire function e^x (see (1.10)), as the following theorem shows.

Theorem 1.28 ([103, p. 343])**.** *Let $T = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence. If $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{L} - \mathcal{P}$, then the function $T[f(x)] := \sum_{k=0}^{\infty} \gamma_k a_k x^k$ represents an entire function, and also belongs to the Laguerre-Pólya class.*

Proof. See the Appendix (Theorem 8.10). □

Definition 1.29. A sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ is called a *complex zero decreasing sequence* (or CZDS, an acronym that will be used in the sequel) if, for any real polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$,

$$Z_c(T[f(x)]) = Z_c\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \le Z_c(f(x)),$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

The following is a special case of Problem 1.10.

Problem 1.30. Characterize all complex zero decreasing sequences.

Every CZDS is necessarily a multiplier sequence, but the converse is not true in general, as the following example shows.

Example 1.31. Let $T = \{1 + k + k^2\}_{k=0}^{\infty}$. We observe that $T[x^n] = (1 + n + n^2)(x^n) = (1 + 2xD + x^2D^2)[x^n]$, and we may represent T as the linear operator $1 + 2xD + x^2D^2$. To show that T is a multiplier sequence, we use the Algebraic Characterization in Theorem 1.27. Equation (1.11) becomes

$$T[(1+x)^n] = (1+x)^{n-2}((n^2+n+1)x^2 + 2(n+1)x + 1). \tag{1.12}$$

The discriminant of the quadratic on the right-hand side of (1.12) is $4n > 0$, and thus, $T[(1+x)^n] \in \mathcal{L}-\mathcal{P}$. Moreover, $\sqrt{4n} < 2(n+1)$, and hence both zeros of the quadratic are negative. Therefore, $T[(1+x)^n] \in \mathcal{L}-\mathcal{P}I$ and by Theorem 1.27, T is a multiplier sequence. We next show, by means of a concrete example, that T is *not* a complex zero decreasing sequence. The polynomial

$$f(x) = (x^2+3x+2)^6(x^2+1)$$

has two non-real zeros, while the polynomial

$$T[f](x) = 64(1+2x)^4(1+28x+179x^2+100x^3+272x^4)$$

has four non-real zeros.

Problem 1.32. It is known (see [42]) that the sequence $\{e^{-k^p}\}_{k=1}^{\infty}$ is a multiplier sequence for integers $k \geq 3$. Are these sequences CZDS (cf. Problem 1.30)?

Problem 1.33. Characterize all multiplier sequences that are not CZDS (cf. Problem 1.30).

Remark 1.34 ([42])**.** If the sequence $T = \{\gamma_k\}_{k=0}^{\infty}$ has no zero terms, then $T[f]$ has the same degree as f, and thus the statement that $T[f]$ has no more non-real zeros than f is equivalent to the statement that $T[f]$ has no fewer real zeros than f. But if T does have terms that are zero, then this is no longer true since $T[f]$ may well have fewer real zeros than f. It is for this reason that we count non-real zeros. The existence of a non-trivial CZDS is assured by the following theorem of Laguerre. Laguerre's theorem was later extended by Pólya (see [119] or [122, pp. 314–321]), and we will prove it in Section 4.

Theorem 1.35 (Laguerre [111, Satz 3.2])**.**

(i) Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitrary real polynomial of degree n, and let $h(x)$ be a polynomial with only real zeros, none of which lie in the interval $(0,n)$. Then, $Z_c(\sum_{k=0}^{n} h(k)a_k x^k) \leq Z_c(f(x))$.

(ii) Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitrary real polynomial of degree n, let $\varphi(x) \in \mathcal{L}-\mathcal{P}$ and suppose that none of the zeros of φ lie in the interval $(0,n)$. Then, $Z_c(\sum_{k=0}^{n} \varphi(k)a_k x^k) \leq Z_c(f(x))$.

(iii) If $\varphi \in \mathcal{L}-\mathcal{P}(-\infty,0]$, then the sequence $\{\varphi(k)\}_{k=0}^{\infty}$ is a CZDS.

Proof. See Section 4. □

In [42], Craven and Csordas characterized all complex zero decreasing sequences of the form $\{h(k)\}_{k=0}^{\infty}$, where $h(x) \in \mathbb{R}[x]$.

Theorem 1.36 ([42, Theorem 2.13])**.** *Let $h(x)$ be a real polynomial. The sequence $T = \{h(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence if and only if either*

(1) $h(0) \neq 0$ and all the zeros of h are real and negative, or

(2) $h(0) = 0$ and the polynomial $h(x)$ is of the form

$$h(x) = x(x-1)(x-2)\cdots(x-m+1)\prod_{i=1}^{p}(x-b_i),$$

where $b_i < m$ for each $i = 1, \ldots, p$.

It turns out that the representation of the linear operator $T = \{\gamma_k\}_{k=0}^{\infty}$ as a formal differential operator (see Theorem 1.14) is tractable, with the aid of the *Appell polynomials*, which we define below.

Definition 1.37. Let $f(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k$ be an entire function. Then the n^{th} *Appell polynomial* associated with f is defined by

$$P_n(t) := \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \gamma_k t^{n-k} \quad (n = 0, 1, 2, \ldots).$$

Remark 1.38. We remark that the Appell polynomials are generated by $e^{xt} f(x) = \sum_{n=0}^{\infty} P_n(t) x^n$ (see the Appendix, Proposition 8.3).

Proposition 1.39. *Let $T = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers, and let*

$$g_n^*(x) = n! P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^{n-k} \quad (n = 0, 1, 2, \ldots).$$

Then the linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ can be represented as

$$T = \sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k D^k,$$

where D denotes differentiation with respect to x.

Proof. Let $\hat{T} = \sum_{k=0}^{\infty} \frac{g_k^*(-1)}{k!} x^k D^k$. To show that $\hat{T} = T$, it is sufficient to verify that $\hat{T}[x^n] = \gamma_n x^n$ for all n. To this end, we calculate

$$\begin{aligned}
\hat{T}[x^n] &= \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{g_k^*(-1)}{k!} x^n = \sum_{k=0}^{n} \binom{n}{k} g_k^*(-1) x^n \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \gamma_j \right) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (-1)^{k-j} \gamma_j.
\end{aligned}$$

Interchanging the order of summation yields

$$\begin{aligned}
\sum_{j=0}^{n} \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} (-1)^{k-j} \gamma_j &= \sum_{j=0}^{n} \frac{\gamma_j}{j!} \sum_{k=j}^{n} \binom{n}{k} \binom{k}{j} (-1)^{k-j} \gamma_j \\
&= \sum_{j=0}^{n} \frac{\gamma_j}{j!} \left[D^j [(1+x)^n] \right]_{x=-1} = \gamma_n.
\end{aligned}$$

□

The Appell polynomials enjoy two important properties: (i) $P_n'(t) = P_{n-1}(t)$ and (ii) if none of the γ_k in (1.11) are zero, then $n! P_n(x)$ is the *reverse* (see Definition 1.40) of the n^{th} Jensen polynomial, $g_n(x)$ (cf. Remark 1.22).

Definition 1.40. Let $f(x) := \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$, $a_n \neq 0$. We define the *reverse* of $f(x)$ to be the polynomial $f^*(x) := x^n f(\frac{1}{x})$ and observe that if f is not a monomial, then $f \in \mathcal{L} - \mathcal{P}$ if and only if $f^* \in \mathcal{L} - \mathcal{P}$.

2. Extension of Linear Operators to a Class of Transcendental Entire Functions

The principal goal of this section is to establish results that extend the action of linear operators from the vector space of polynomials to the transcendental entire functions in the Laguerre-Pólya class and the associated $\mathcal{L}-\mathcal{P}^*$ class (see Definition 1.21). We commence by proving some intrinsic properties of functions in the Laguerre-Pólya class (see Theorem 2.2, Theorem 2.3 and Corollary 2.4). We subsequently prove that when the linear operator T is represented by a finite order differential operator (cf. Theorem 1.14), it preserves uniform convergence of sequences of entire functions when T is extended to the transcendental functions in the Laguerre-Pólya class (see Theorem 2.5). If, in addition, T is also a complex zero decreasing operator on the vector space of polynomials, then we prove that T is a complex zero decreasing operator on the class $\mathcal{L}-\mathcal{P}^*$ (cf. Theorem 2.8).

Definition 2.1 ([40])**.** If

$$f(x)=\sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k \quad (\gamma_k\in\mathbb{R}, k=0,1,2,\ldots)$$

is a real entire function, then the n^{th} *Jensen polynomial* associated with $f(x)$ is defined by

$$g_n(x):=\sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k \quad (n=0,1,2,\ldots). \tag{2.1}$$

The n^{th} Jensen polynomial associated with $f^{(p)}(x)$, $p=0,1,2,\ldots$, is denoted by

$$g_{n,p}(x):=\sum_{k=0}^{n}\binom{n}{k}\gamma_{k+p} x^k \quad (n,p=0,1,2,\ldots). \tag{2.2}$$

The Jensen polynomials associated with a given entire function satisfy several important properties (cf. [40]). In particular, they can be used to approximate entire functions, as the following theorem shows.

Theorem 2.2 (cf. [40, Lemma 2.2])**.** *Let*

$$h(z)=\sum_{k=0}^{\infty}\frac{a_k}{k!}z^k \quad (a_k\in\mathbb{C})$$

be an arbitrary entire function. For each fixed non-negative integer p, let

$$g_{n,p}(z)=\sum_{k=0}^{n}\binom{n}{k}a_{k+p} z^k \quad (n=0,1,2,\ldots).$$

Then,

$$\lim_{n\to\infty} g_{n,p}\left(\frac{z}{n}\right)=h^{(p)}(z) \quad (p=0,1,2,\ldots) \tag{2.3}$$

uniformly on compact subsets of $\mathbb{C}$.

Proof. For a fixed non-negative integer p and $n \geq 2$,

$$g_{n,p}\left(\frac{z}{n}\right) = a_p + a_{p+1}z + \sum_{k=2}^{n}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{a_{k+p}}{k!}z^k.$$

Then, by the Cauchy inequalities for the Taylor coefficients of $h^{(p)}(z)$, we have

$$\left|\frac{a_{k+p}}{k!}\right| \leq \frac{M(R,h^{(p)})}{R^k} \quad (k = 1,2,3,\ldots, R > 0),$$

where

$$M(R,h^{(p)}) = \max_{|z|=R}|h^{(p)}(z)|.$$

Thus, for $n > m+1$ and for $0 < |z| \leq r < R = 2r$, we obtain

$$\left|\sum_{k=m+1}^{n}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{a_{k+p}}{k!}z^k\right| \\ \leq M(R,h^{(p)})\frac{r^{m+1}}{R^m(R-r)}, \tag{2.4}$$

and

$$\left|\sum_{k=m+1}^{\infty}\frac{a_{k+p}}{k!}z^k\right| \leq M(R,h^{(p)})\frac{r^{m+1}}{R^m(R-r)}. \tag{2.5}$$

Therefore, with $R = 2r$ and for any $\epsilon > 0$, there is a positive integer m_0 such that $r^{m+1}(R^m(R-r))^{-1} < \epsilon$ for all $m \geq m_0$. Finally, there is a positive integer $N > m_0$ such that for all $n \geq N$ and $|z| \leq r$,

$$\left|\sum_{k=2}^{m_0}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{a_{k+p}}{k!}z^k - \sum_{k=2}^{m_0}\frac{a_{k+p}}{k!}z^k\right| < \epsilon, \tag{2.6}$$

and hence (2.3) follows from (2.4), (2.5) and (2.6).

□

We recall that in Remark 1.22, we stated several useful properties of functions in the Laguerre-Pólya class. As a consequence of Theorem 2.2, we are now able to establish some of these properties.

Theorem 2.3. *If $f(x) \in \mathcal{L}-\mathcal{P}$, then the associated Jensen polynomials, $g_n(x)$, are hyperbolic.*

Proof. Let $f(x) = \sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k$ be a function in the Laguerre-Pólya class. Then by Theorem 3.10, the linear operator $f(D) : \mathbb{R}[x] \to \mathbb{R}[x]$ (cf. (1.1)) is a hyperbolicity preserver. Thus, for all $n \in \mathbb{N}$, the polynomial

$$f(D)\left[\frac{x^n}{n!}\right] = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\gamma_k x^{n-k} = P_n(x)$$

is hyperbolic, where $P_n(x)$ is the n^{th} Appell polynomial associated with $f(x)$ (see Definition 1.37). The reverse of $P_n(x)$ (cf. Definition 1.40) is

$$n!x^n P_n\left(\frac{1}{x}\right) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k = g_n(x).$$

By the observation in Definition 1.40, it follows that $g_n(x)$ is hyperbolic for all n. $\square$

Corollary 2.4. *If $f(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k \in \mathcal{L}-\mathcal{P}$, then $f'(x) \in \mathcal{L}-\mathcal{P}$. That is, $\mathcal{L}-\mathcal{P}$ is closed under differentiation.*

Proof. Since $f(x)$ is in the Laguerre-Pólya class, we know by Theorem 2.3 that its associated Jensen polynomials, $g_n(x)$, are hyperbolic for each n. Now, taking the derivative of $g_n(x)$, we get

$$g_n'(x) := \sum_{k=0}^{n-1} (k+1) \binom{n}{k+1} \gamma_{k+1} x^k = n g_{n-1,1}(x) \quad (n = 0, 1, 2, \ldots). \tag{2.7}$$

Since derivative of a hyperbolic polynomial is always hyperbolic, the polynomial $g_{n,1}(x)$ is hyperbolic for each n. By Theorem 2.2, $g_{n,1}(\frac{x}{n}) \to f'(x)$ uniformly on compact subsets of $\mathbb{C}$, and whence we conclude that $f' \in \mathcal{L}-\mathcal{P}$ (cf. Remark 1.22). $\square$

We next consider the action of linear operators T on the transcendental entire functions in the class $\mathcal{L}-\mathcal{P}^*$. We recall that by Theorem 1.14, every linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ can be expressed as a formal differential operator with real polynomial coefficients. However, in general, the issue of convergence often arises in applying such linear operators to transcendental entire functions (see the example in (1.1a)). Moreover, if $T = \sum_{k=0}^{\infty} D^k$ is a linear operator on the space of entire functions, then for $r > 1$,

$$T[e^{rx}] = \left(\sum_{k=0}^{\infty} D^k\right) [e^{rx}] = e^{rx} \sum_{k=0}^{\infty} r^k;$$

the series does not converge for any x.

Of course, the issue of convergence disappears if we restrict our considerations to differential operators of finite order. The following theorem, which is more general than we need, demonstrates that differential operators of finite order preserve uniform convergence.

Theorem 2.5 (cf. [116, p. 62])**.** *Let T be a linear operator of the form*

$$T = \sum_{k=0}^{m} T_k(z) D^k, \tag{2.8}$$

where each $T_k(z)$ is an entire function and $D = d/dz$. If the sequence of entire functions $\{f_n(z)\}_{n=1}^{\infty}$ converges to the entire function $f(z)$ uniformly on compact subsets of $\mathbb{C}$, then the sequence of entire functions $\{T[f_n](z)\}_{n=1}^{\infty}$ converges to the function $T[f](z)$ uniformly on compact subsets of $\mathbb{C}$.

Proof. Let K be a compact subset of $\mathbb{C}$ and suppose $\epsilon > 0$. Since each of the functions T_k are entire, there exists a constant M such that $|T_k(z)| \leq M$ for all $k = 1, 2, \ldots, m$ and for all $z \in K$. Since $f_n \to f$ on compact subsets of $\mathbb{C}$, $f_n^{(k)} \to f^{(k)}$

uniformly on compact subsets of $\mathbb{C}$. Thus, we can pick an integer N such that, for all $k = 1, 2, \ldots, m$,

$$|f_n^{(k)}(z) - f^{(k)}(z)| < \frac{\epsilon}{(m+1)M},$$

whenever $n \geq N$ and $z \in K$. A calculation shows that whenever $n \geq N$ and $z \in K$,

$$\begin{aligned}
|T[f_n](z) - T[f](z)| &= \left| \sum_{k=0}^{m} T_k(z) f_n^{(k)}(z) - \sum_{k=0}^{m} T_k(z) f^{(k)}(z) \right| \\
&= \left| \sum_{k=0}^{m} T_k(z)(f_n^{(k)}(z) - f^{(k)}(z)) \right| \\
&\leq \sum_{k=0}^{m} |T_k(z)||(f_n^{(k)}(z) - f^{(k)}(z))| \\
&\leq \sum_{k=0}^{m} M \cdot \frac{\epsilon}{(1+m)M} = \epsilon.
\end{aligned}$$

□

The hypothesis of uniform convergence is fundamental. In the sequel, we will apply Hurwitz's Theorem to obtain results which extends certain types of linear operators to the functions in the class $\mathcal{L}-\mathcal{P}$ and its associated class $\mathcal{L}-\mathcal{P}^*$.

Corollary 2.6. *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a hyperbolicity preserver (cf. Definition 1.5) and suppose that T can be expressed in the form*

$$T = \sum_{k=0}^{m} Q_k(x) D^k, \tag{2.9}$$

where $Q_k(x) \in \mathbb{R}[x]$. If $f(x) \in \mathcal{L}-\mathcal{P}$, then $T[f](x) \in \mathcal{L}-\mathcal{P}$.

Proof. Since $f(x) \in \mathcal{L}-\mathcal{P}$, the Jensen polynomials, $g_n(x)$, associated with $f(x)$ are hyperbolic and approximate $f(x)$ uniformly on compact subsets of $\mathbb{C}$ (cf. Theorems 2.2 and 2.3). By Theorem 2.5, $T[g_n(x)] \to T[f(x)]$ uniformly on compact subsets of $\mathbb{C}$. However, as each $g_n(x)$ is a polynomial and T is a hyperbolicity preserver, we know that $T[g_n(x)] \in \mathcal{L}-\mathcal{P}$ for each n. Since $T[f(x)]$ can be approximated uniformly on compact subsets of $\mathbb{C}$ by a sequence of hyperbolic polynomials, by Remark 1.22 we can conclude that $T[f(x)] \in \mathcal{L}-\mathcal{P}$. □

Remark 2.7. We observe that $\mathcal{L}-\mathcal{P}^*$ is closed under differentiation. Let $f(x) = p(x)\psi(x) \in \mathcal{L}-\mathcal{P}^*$, where $p(x)$ is a real polynomial with $2N$ non-real zeros and $\psi(x) \in \mathcal{L}-\mathcal{P}$. By Theorem 2.2 and Theorem 2.3, the Jensen polynomials, $\psi_n(x)$, associated with $\psi(x)$ are hyperbolic and converge to $\psi(x)$ uniformly on compact subsets of $\mathbb{C}$. Let $f_n(x) = p(x)\psi_n(x)$. Then $f_n(x) \to f(x)$ uniformly on compact subsets of $\mathbb{C}$, and thus, $f_n'(x) \to f'(x)$ uniformly on compact subsets of $\mathbb{C}$. By Rolle's Theorem, $Z_c(f_n'(x)) \leq Z_c(f_n(x))$ for each n, and consequently, by Hurwitz's Theorem, $Z_c(f'(x)) \leq Z_c(f(x))$. Since $f(x) \in \mathcal{L}-\mathcal{P}$, we can express it in the form $f(x) = e^{-\alpha_1 x^2} f_1(x)$, where $f_1(x)$ is of genus 0 or 1. If we write $f'(x) = e^{-\alpha_2 x^2} f_2(x)$, where $f_2(x) \in \mathcal{L}-\mathcal{P}^*$ is of genus 0 or 1, then an argument similar to the one used by Pólya and Schur [123, p. 109] shows that $\alpha_2 \geq \alpha_1$, and hence, $f'(x) \in \mathcal{L}-\mathcal{P}^*$.

Theorem 2.8. *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a complex zero decreasing operator and suppose that T can be expressed in the form*

$$T = \sum_{k=0}^{m} Q_k(x) D^k,$$

where $Q_k(x) \in \mathbb{R}[x]$. Let $f(x) \in \mathcal{L} - \mathcal{P}^*$, where $f(x) = p(x)\varphi(x)$ with $p(x) \in \mathbb{R}[x]$ and $\varphi(x) \in \mathcal{L} - \mathcal{P}$ (cf. Definition 1.21). Then, $T[f](x)$ has at most a finite number of non-real zeros, and

$$Z_c(T[f](x)) \leq Z_c(f(x)),$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities. In addition, if the order of $f(x)$, $\rho(f(x))$, is strictly less than 2, then $T[f(x)] \in \mathcal{L} - \mathcal{P}^*$.

Proof. Suppose that $f(x) = p(x)\varphi(x)$, where $p(x) \in \mathbb{R}[x]$ has exactly $2N$ non-real zeros and $\varphi(x)$ is in the Laguerre-Pólya class. Then the Jensen polynomials, $g_n(x)$, associated with $\varphi(x)$ are hyperbolic and approximate $\varphi(x)$ uniformly on compact subsets of $\mathbb{C}$ (see Theorems 2.2 and 2.3). Consequently, $Z_c(g_n(x)p(x)) = 2N$ and $g_n(x)p(x) \to \varphi(x)p(x) = f(x)$ uniformly on compact subsets of $\mathbb{C}$. By Theorem 2.5, $T[g_n(x)p(x)] \to T[\varphi(x)p(x)]$ uniformly on compact subsets of $\mathbb{C}$. By Hurwitz's Theorem, for n sufficiently large, $Z_c(T[f](x)) \leq Z_c(T[g_n(x)p(x)])$ (see the Appendix, Corollary 8.6). Since T is a complex zero decreasing operator, for each $n \in \mathbb{N}$,

$$Z_c(T[g_n(x)p(x)]) \leq Z_c(g_n(x)p(x)) = 2N = Z_c(f(x)).$$

Thus,

$$Z_c(T[f(x)]) \leq Z_c(f(x)). \tag{2.10}$$

Now, suppose that $\rho(f(x)) < 2$. By Remark 2.6, for each $k \leq m$, $Q_k(x) f^{(k)}(x) \in \mathcal{L} - \mathcal{P}^*$. Furthermore, $\rho(f^{(k)}(x)) = \rho(f(x)) < 2$ for each k (cf. [19, p. 13, Theorem 2.4.1]), and thus,

$$\rho(T[f(x)]) = \rho\left(\sum_{k=0}^{m} Q_k(x) f^{(k)}(x)\right) < 2.$$

By (2.10), $T[f(x)]$ has a finite number of non-real zeros $z_1, z_2, \ldots, z_r$, where $r = Z_c(T[f(x)])$. We may then express $T[f(x)]$ in the form $T[f(x)] = q(x)h(x)$, where

$$q(x) = \prod_{k=0}^{r} (z - z_k)$$

and $h(x)$ has no non-real zeros. Since $\rho(h(x)) < 2$, it follows that $h(x) \in \mathcal{L} - \mathcal{P}$. Thus, $T[f(x)] \in \mathcal{L} - \mathcal{P}^*$. □

Example 2.9. It is not true, in general, that if T is a hyperbolicity preserver of the form in (2.9), then $T[f](x)$ has only real zeros whenever $f(x)$ is an entire function with only real zeros (cf. Corollary 2.6). For example, if $T = D = d/dx$ and $f(x) = e^{x^3+3x}$, then $T[f](x) = 3(x^2+1)e^{x^3+3x}$, which has two non-real zeros.

3. Generalizations of the Hermite-Poulain Theorem

In an attempt to generalize the Hermite-Poulain Theorem (Theorem 1.12), the natural question arises if these results remain valid when one replaces the differential operator, $D = d/dx$, with another type of linear operator. The familiar proofs of the aforementioned Hermite-Poulain Theorem hinge on Rolle's Theorem, which says that $D[f]$ has an odd number of real zeros between any two consecutive real zeros of f. It is for this reason that the main theorem in this section (see Corollary 3.6 and Corollary 3.7) generalizes the Hermite-Poulain Theorem by replacing the differential operator, $D = d/dx$, with a linear operator T that has the *strict interlacing property* (see Definition 3.2). In this section, we will also generalize Theorem 1.12 to the $\mathcal{L} - \mathcal{P}^*$ class of entire functions (cf. Definition 1.21, Theorem 3.9 and Theorem 3.10). We begin this section by introducing the definition of interlacing zeros. The notion of interlacing zeros is the subject of S. Fisk's recent monumental tome (cf. [64, Chapter 1]).

Definition 3.1. Let $f(x), g(x) \in \mathbb{R}[x]$ be hyperbolic polynomials with $|\deg f - \deg g| \leq 1$, and suppose that there exists an ordering

$$x_1 \leq y_1 \leq x_2 \leq y_2 \leq \dots, \tag{3.1}$$

where the x_j's and the y_j's denote the zeros of these polynomials. Then we say that f and g have *weakly interlacing zeros*. If the inequalities in (3.1) are strict inequalities, then we say that f and g have *strictly interlacing zeros*.

Definition 3.2. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is said to possess the *weak interlacing property* if, whenever $f(x)$ is a hyperbolic polynomial, then $T[f]$ and f have weakly interlacing zeros. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is said to possess the *strict interlacing property* if, whenever $f(x)$ additionally has *simple* zeros, then $T[f]$ and f have strictly interlacing zeros. Note that if T possesses the weak interlacing property, then it must necessarily be a hyperbolicity preserver.

The following theorem of Hermite and Biehler is remarkable in that it gives a necessary and sufficient condition for two polynomials to have strictly interlacing zeros. We will later make use of this theorem to generalize the Hermite-Poulain Theorem.

Theorem 3.3 (Hermite-Biehler [103, Chapter 7])**.** *Let*

$$f(z) = p(z) + iq(z) = c\prod_{k=1}^{n}(z - \alpha_k) \quad (0 \neq c \in \mathbb{R}),$$

where $p(z), q(z)$ are real polynomials of degree at least 2. Then $p(z)$ and $q(z)$ have strictly interlacing zeros if and only if the zeros of $f(z)$ are all located in either the open upper half-plane or the open lower half-plane.

To prove Theorem 3.3, we first establish the following proposition.

Proposition 3.4. *Let*

$$f(z) = p(z) + iq(z) = c\prod_{k=1}^{n}(z - \alpha_k) \quad (0 \neq c \in \mathbb{R}), \tag{3.2}$$

where $p(z), q(z)$ are real polynomials of degree at least 2.

1. If $\mathrm{Im}(\alpha_k) > 0$ *for all* k*, then*

(a) p *and* q *have strictly interlacing zeros, and*

(b) for all real x*,*

$$d(x) := q'(x)p(x) - q(x)p'(x) > 0. \tag{3.3}$$

2. If p *and* q *have strictly interlacing zeros, and for some* x_0*,* $q'(x_0)p(x_0) - q(x_0)p'(x_0) > 0$*, then all of the zeros of* f *are in Im* $z > 0$*.*

Proof. 1. Since p and q are real polynomials,

$$\overline{p(\overline{z}) + iq(\overline{z})} = c\overline{\prod_{k=1}^{n}(\overline{z} - \alpha_k)} = c\prod_{k=1}^{n}(z - \overline{a_k}) = p(z) - iq(z). \tag{3.4}$$

Let $p(z_0) = 0$. Then, $p(\overline{z_0}) = 0$ and $f(z_0) = \overline{iq(\overline{z_0})} = -iq(z_0)$, and consequently, $q(\overline{z_0}) = q(z_0)$. Thus, $f(\overline{z_0}) = f(z_0)$, and

$$|(z_0 - \alpha_1)(z_0 - \alpha_2)\cdots(z_0 - \alpha_n)| = |(z_0 - \overline{\alpha_1})(z_0 - \overline{\alpha_2})\cdots(z_0 - \overline{\alpha_n})|. \tag{3.5}$$

Since $\mathrm{Im}\,\alpha_k > 0$ for all k, if $\mathrm{Im}\,z_0 > 0$, then $|z_0 - \alpha_k| < |z_0 - \overline{\alpha_k}|$ for all k, invalidating (3.5). Similarly, if $\mathrm{Im}\,z_0 < 0$, then $|z_0 - \alpha_k| > |z_0 - \overline{\alpha_k}|$ for all k and again (3.5) fails. Thus, $\mathrm{Im}\,z_0 = 0$, which means that every zero of p is real.

For real values λ and μ, consider the polynomial

$$(\lambda - i\mu)(p(z) + iq(z)) = \lambda p(z) + \mu q(z) + i(\lambda q(z) - \mu p(z)),$$

all of whose zeros lie in $\mathrm{Im}\,z > 0$. Then by the above argument the polynomial $\lambda p(z) + \mu q(z)$ has only real zeros. Thus, with $\lambda = 0$, we see that $q(z)$ also has only real zeros. Moreover, $p(z)$ and $q(z)$ have no common zeros, for otherwise the conjugate of that zero will also be a zero of f, violating the fact that the zeros of $f(z)$ all lie in the open upper-half plane $\mathrm{Im}\,z > 0$. In order to show that the zeros of $p(z)$ and $q(z)$ are simple and interlace, we calculate the imaginary part of the logarithmic derivative of $f(z)$. For $x \in \mathbb{R}$,

$$\mathrm{Im}\,\frac{f'(x)}{f(x)} = \sum_{k=1}^{n}\mathrm{Im}\,\frac{1}{x - \alpha_k} = \sum_{k=1}^{n}\mathrm{Im}\,\frac{\alpha_k}{|x - \alpha_k|^2} > 0. \tag{3.6}$$

On the other hand, by (3.6),

$$\mathrm{Im}\,\frac{f'(x)}{f(x)} = \mathrm{Im}\,\frac{p'(x) + iq'(x)}{p(x) + iq(x)} = \frac{q'(x)p(x) - p'(x)q(x)}{|p(x) + iq(x)|^2} > 0. \tag{3.7}$$

Thus, we see (by (3.7)) that $d(x) = q'(x)p(x) - q(x)p'(x) > 0$ (cf. (3.3)), and hence the zeros of both p and q are simple. Next, let t_k and t_{k+1} denote two consecutive zeros of p. Then

$$d(t_k) = -p'(t_k)q(t_k) > 0 \text{ and } d(t_{k+1}) = -p'(t_{k+1})q(t_{k+1}) > 0,$$

thus,

$$p'(t_k)q(t_k)p'(t_{k+1})q(t_{k+1}) > 0. \tag{3.8}$$

Since the zeros of p are all simple, $p'(t_k)p'(t_{k+1}) < 0$. Therefore, it follows from (3.8) that $q(t_k)q(t_{k+1}) < 0$. Hence, q vanishes between t_k and t_{k+1}. A similar argument (using the fact that $d(x) > 0$ for $x \in \mathbb{R}$) shows that p vanishes between two consecutive zeros of q. (In particular, it follows that $|\deg p - \deg q| \leq 1$.)

2. To prove Part 2, suppose that p and q have only simple, real zeros which interlace and that for some real x_0, $q'(x_0)p(x_0) - q(x_0)p'(x_0) > 0$. We first note that a zero, α_k of f, cannot be real (see (3.2)). (Indeed, suppose that α_k is real. Then $0 = f(\alpha_k) = p(\alpha_k) + iq(\alpha_k) \Rightarrow p(\alpha_k) = 0$ and $q(\alpha_k) = 0$. But this contradicts the assumption that the zeros of p and q are interlacing). We can assume without loss of generality that $\deg p \geq \deg q$ and that p is monic. Let $t_1, \ldots, t_n$ denote the zeros of p, such that $p(z) = \prod_{k=1}^n (z - t_k)$. Then, $p'(z) = (\prod_{k=1}^n (z - t_k))' = \sum_{k=1}^n \prod_{j \neq k} (z - t_j)$. Evaluated at t_k, this becomes $p'(t_k) = \prod_{j \neq k} (t_k - t_j)$. (*) We know that $n - 1 \leq \deg q \leq n$, and we may express $q(z)$ as $q(z) = \sum_{k=0}^n a_k z^k$, where a_n may be zero. Let $c = -a_n$ and set $q_1(z) = cp(z) + q(z)$. Since $p(z)$ is monic, $q_1(z)$ has a zero n^{th}-degree term, and thus $\deg q_1(z) \leq n - 1$. Of course, if $\deg q = n - 1$, then $c = 0$ and $q_1(z) = q(z)$.

Now, let $h(z) = p(z)(\sum_{k=1}^n \frac{q_1(t_k)}{p'(t_k)(z - t_k)})$. With (*), a calculation shows that

$$h(z) = \sum_{k=1}^n \frac{q_1(t_k)}{p'(t_k)} \frac{p(z)}{(z - t_k)} = \sum_{k=1}^n q_1(t_k) \prod_{j \neq k} \frac{z - t_j}{t_k - t_j}.$$

We see that $h(z)$ is a polynomial of degree no more than $n-1$. Also, when evaluated at t_m $(m = 1, 2, \ldots, n)$,

$$h(t_m) = q_1(t_m) \prod_{j \neq m} \frac{t_m - t_j}{t_m - t_j} = q_1(t_m),$$

and whence h agrees with q_1 on at least n distinct points. Since polynomials of degree $n - 1$ (or less) are determined by its values at n distinct points (see the Appendix, Proposition 8.4), it follows that $h(z) = q_1(z)$ and

$$\frac{q_1(z)}{p(z)} = \sum_{k=1}^n \frac{q_1(t_k)}{p'(t_k)(z - t_k)}.$$

However, recall that $q(z) = -cp(z) + q_1(z)$, and thus $q_1(z)/p(z) = q(z)/p(z) + c$. Also, when $q(z)$ is evaluated at t_k, $q(t_k) = -cp(t_k) + q_1(t_k) = q_1(t_k)$. Therefore,

$$\frac{q(z)}{p(z)} + c = \frac{q_1(z)}{p(z)} = \sum_{k=1}^n \frac{q(t_k)}{p'(t_k)(z - t_k)}. \tag{3.9}$$

Since the zeros of p and q are interlacing, $q(t_k)q(t_{k+1}) < 0$, and by Rolle's theorem, $p'(t_k)p'(t_{k+1}) < 0$. Thus, the numbers $\frac{q(t_k)}{p'(t_k)}$ have the same sign for all t_k. Now, at a zero α_j of f, $0 = f(\alpha_j) = p(\alpha_j) + iq(\alpha_j)$ and a calculation shows that (for any α_j),

$$\operatorname{Im}\left(\frac{q(\alpha_j)}{p(\alpha_j)}\right) = 1.$$

Since c is a real number,

$$\operatorname{Im}\left(\frac{q(\alpha_j)}{p(\alpha_j)}\right) = \operatorname{Im}\left(\frac{q(\alpha_j)}{p(\alpha_j)} + c\right),$$

and by (3.9),

$$1 = \operatorname{Im}\left(\sum_{k=1}^{n} \frac{q(t_k)}{p'(t_k)} \frac{1}{\alpha_j - t_k}\right) = -(\operatorname{Im}\alpha_j) \sum_{k=1}^{n} \frac{q(t_k)}{p'(t_k)} \frac{1}{|\alpha_j - t_k|^2}. \tag{3.10}$$

Since all the numbers $\frac{q(t_k)}{p'(t_k)}$ have the same sign, it follows from (3.10) that the $\operatorname{Im}\alpha_j$ all have the same sign. We next show that the assumption that for some real $x_0, q'(x_0)p(x_0) - q(x_0)p'(x_0) > 0$, implies that $\operatorname{Im}\alpha_j > 0$. We have

$$\begin{aligned}\operatorname{Im}\frac{f'(x_0)}{f(x_0)} &= \sum_{k=1}^{n} \frac{\operatorname{Im}\alpha_k}{|x_0 - \alpha_k|^2} \\ &= \operatorname{Im}\frac{p'(x_0) + iq'(x_0)}{p(x_0) + iq(x_0)} = \operatorname{Im}\frac{q'(x_0)p(x_0) - p'(x_0)q(x_0)}{|f(x_0)|^2} > 0,\end{aligned}$$

and thus, $\operatorname{Im}\alpha_k > 0$ for all $k, k = 1, 2 \ldots, n$. □

By an argument of reversing signs in the proof of Proposition 3.4, we can show that (1) if $\operatorname{Im}(\alpha_k) < 0$ for all k, then p and q have strictly interlacing zeros and for all real x, $d(x) := q'(x)p(x) - q(x)p'(x) < 0$, and (2) if p and q have strictly interlacing zeros and for some x_0, $q'(x_0)p(x_0) - q(x_0)p'(x_0) < 0$, then all of the zeros of f are in $\operatorname{Im} z < 0$. This together with Proposition 3.4 gives us Theorem 3.3. We now state a necessary and sufficient condition for two polynomials to have strictly interlacing zeros, which we will prove by using the Hermite-Biehler Theorem (Theorem 3.3).

Theorem 3.5 ([111, p. 13], [103, p. 305])**.** *Let f, g be real polynomials. The polynomial $\alpha f + \beta g$ has only real, simple zeros for all $\alpha, \beta \in \mathbb{R}$ $(\alpha^2 + \beta^2 \neq 0)$ if and only if f and g have strictly interlacing zeros.*

Proof. Suppose that f and g have strictly interlacing zeros. By the Hermite-Biehler Theorem, the zeros of $f + ig$ all lie on either the upper open half-plane or the lower open half-plane. Thus, $(\alpha - i\beta)(f + ig) = (\alpha f + \beta g) + i(\alpha f - \beta g)$ has only zeros that all lie on either the open upper half-plane or the open lower half-plane. Again by the Hermite-Biehler Theorem, $\alpha f + \beta g$ and $\alpha f - \beta g$ have strictly interlacing zeros. More to the point, $\alpha f + \beta g$ has only real, simple zeros.

Now, suppose conversely that $\alpha f + \beta g$ has only simple zeros for all $\alpha, \beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$. Then f and g must both have only real, simple zeros, as we can set $\alpha = 1, \beta = 0$ or $\alpha = 0, \beta = 1$. Secondly, without loss of generality, let $\deg f \geq \deg g = n$. Since $f + cg$ has only real zeros, it must be that $f^{(n)} + cg^{(n)}$ has only real zeros for all c, making $f^{(n)}$ necessarily of first degree or less. Therefore, the degrees of f and g differ by at most one. Thirdly, suppose that a point y is a zero of both f and g. Then we can pick non-zero α, β such that $\alpha f'(y) = -\beta g'(y)$.

However, $\alpha f + \beta g$ has a multiple zero at y, a contradiction. Thus, f and g cannot share a zero.

Finally, to prove that the zeros of f and g strictly interlace, we note that the roots of $\frac{f(x)}{g(x)} - \lambda = 0$ are the zeros of $f(x) - \lambda g(x)$, and thus are simple. The case when $\frac{f(x)}{g(x)} - \lambda$ is evaluated at some x_0, where $g(x_0) = 0$, is not a contradiction since $f(x_0) = g(x_0) = 0$. This is impossible as f and g cannot share zeros. Consequently, $\frac{f(x)}{g(x)}$ is increasing or decreasing on its domain of definition. Thus, $h(x) = (g(x)^2)(\frac{f(x)}{g(x)})' = (f'(x)g(x) - f(x)g'(x))$ always has the same sign for every real x.

Now, let $a < b$ be two consecutive real zeros of f. Then,

$$h(a)h(b) = f'(a)f'(b)g(a)g(b) > 0.$$

But since f has simple zeros, it follows that $f'(a)f'(b) < 0$, and consequently $g(a)g(b) < 0$. Thus, g has a zero in the interval (a, b). Similarly, if $a' < b'$ are two consecutive zeros of g, then

$$h(a')h(b') = f(a')f(b')g'(a')g'(b') > 0,$$

and since $g'(a')g'(b') < 0$, it follows that $f(a')f(b') < 0$. Hence, f has a zero in the interval (a', b'). From this we conclude that f and g have strictly interlacing zeros. □

By replacing g with $T[f]$ in the previous theorem, we can state a corollary about linear operators with the strict interlacing property (cf. Definition 3.2).

Corollary 3.6. *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. If T possesses the strict interlacing property (cf. Definition 3.2), then the linear operator $\alpha + T$ is a hyperbolicity preserver for all $\alpha \in \mathbb{R}$.*

The n-fold iteration gives the following corollary, which is an extension of Hermite-Poulain (cf. Theorem 1.12).

Corollary 3.7. *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. Let $h(y) = \sum_{k=0}^{n} c_k y^k$ be a hyperbolic polynomial. Define the linear operator $h(T) : \mathbb{R}[x] \to \mathbb{R}[x]$ by*

$$h(T)[f(x)] := \sum_{k=0}^{n} c_k T^k[f(x)].$$

If T possesses the strict interlacing property, then $h(T)$ is a hyperbolicity preserver.

Proof. By the Fundamental Theorem of Algebra, $h(y) = c\prod_{k=0}^{n}(y + a_k)$ for some $a_k \in \mathbb{R}$. First, let us prove the theorem in the case where f is hyperbolic and has only simple zeros. Then, f and $T[f]$ have strictly interlacing zeros, and by Theorem 3.5, $a_1 f + T[f] = (a_1 + T)(f)$ has only real, simple zeros. Thus, $a_1 f + T[f]$ and $T[a_1 f + T[f]]$ have strictly interlacing zeros by the property of T, and again by Theorem 3.5, $a_2(a_1 f + T[f]) + T[a_1 f + T[f])] = a_2 a_1 f + (a_2 + a_1)T[f] + T^2[f] = ((a_2 + T)(a_1 + T))[f]$ has only real, simple zeros. Continuing in this fashion, we define

$$h_1(x) = (a_1 + T)f(x) = a_1 f(x) + Tf(x),$$
$$h_2(x) = (a_2 + T)h_1(x) = a_2 a_1 f(x) + (a_2 + a_1)T[f](x) + T^2[f](x),$$
$$\dots h_n(x) = (a_n + T)h_{n-1}(x) = \left(\frac{1}{c}\right) h(T)[f(x)],$$

and we conclude that $h(T)[f(x)]$ has only real, simple zeros.

In the general case, let $f(x) = d\prod_{k=1}^{m}(x - b_k)$ be hyperbolic, possibly having multiple zeros. For $\epsilon > 0$, we define

$$f_\epsilon(x) = d\prod_{k=1}^{m}(x - (b_k + k\epsilon)).$$

For ϵ sufficiently small, this is a hyperbolic polynomial with simple zeros, and $h(T)[f_\epsilon]$ has only real, simple zeros. Since $h(T)[f_\epsilon] \to h(T)[f]$ uniformly as $\epsilon \to 0$, by Hurwitz's Theorem, $h(T)[f]$ is hyperbolic. □

Example 3.8. Corollary 3.6 is an extension of the Hermite-Poulain Theorem (Theorem 1.12) because the differential operator, $D = d/dx$, possesses the strict interlacing property. However, Corollary 3.6 is not a generalization of Laguerre's Theorem (Theorem 1.35), as the operator xD does *not* possess the strict interlacing property. For example, if $f(x) = (x-1)(x+2)$, then $xf'(x) = 2x^2 + x$. The polynomial $f(x)$ has zeros at 1 and -2, while $xD[f(x)]$ has zeros at 0 and $-\frac{1}{2}$. Thus, the two polynomials do not have interlacing zeros.

We now consider the generalization of the Hermite-Poulain Theorem to the case when $f(x) \in \mathcal{L} - \mathcal{P}^*$. This is a simple consequence of Theorem 1.12, Theorem 2.8 and Remark 2.7.

Theorem 3.9. *Let $h(y) = \sum_{k=0}^{n} c_k y^k$ be a real, hyperbolic polynomial. Then, for any function $f(x) \in \mathcal{L} - \mathcal{P}^*$, $h(D)[f(x)] \in \mathcal{L} - \mathcal{P}^*$, and*

$$Z_c(h(D)[f(x)]) = Z_c\left(\sum_{k=0}^{n} c_k f^{(k)}(x)\right) \le Z_c(f(x)),$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

Proof. Let $f(x) \in \mathcal{L} - \mathcal{P}^*$ and let $\alpha \in \mathbb{R}$. By the definition of the Laguerre-Pólya class, we know that $e^{\alpha x} f(x) \in \mathcal{L} - \mathcal{P}^*$. By Remark 2.7, $\mathcal{L} - \mathcal{P}^*$ is closed under differentiation, and since $D[e^{\alpha x} f(x)] = e^{\alpha x}(\alpha f(x) + f'(x))$, it follows that $(D + \alpha)[f] \in \mathcal{L} - \mathcal{P}^*$. By a construction similar to that used in the proof of Theorem 1.12, we obtain that $h(D)[f(x)] \in \mathcal{L} - \mathcal{P}^*$. To show that $Z_c(h(D)[f(x)]) \le Z_c(f(x))$, we note that by Theorem 1.12, $h(D)$ is a complex zero decreasing operator on $\mathbb{R}[x]$. We apply Theorem 2.8 to get that

$$Z_c(h(D)[f](x)) \le Z_c(f(x)),$$

as desired. □

We can thus use Theorem 3.9 to generalize the Hermite-Poulain Theorem even further.

Theorem 3.10 ([41, Lemma 3.1] and [41, Lemma 3.2]). *Let $\varphi(y) = \sum_{k=0}^{\infty} \gamma_k y^k$ be an entire function in the Laguerre-Pólya class. Then the operator*

$$\varphi(D) = \sum_{k=0}^{\infty} \gamma_k D^k$$

has the property that, if $f(x) \in \mathcal{L} - \mathcal{P}^*$ such that $\varphi(D)[f(x)]$ is an entire function, then

$$Z_c(\varphi(D)[f(x)]) \leq Z_c(f(x))$$

Proof. Since $\varphi(x) \in \mathcal{L} - \mathcal{P}$, it can be represented as $\varphi(x) = e^{-\alpha_1 x^2} \varphi_1(x)$, where $\alpha_1 \geq 0$ and $\varphi_1(x)$ is an entire function of genus at most one. Thus, $\varphi(x)$ is an entire function not exceeding the normal type $|\alpha_1|$ of order 2. Similarly, $f(x) \in \mathcal{L} - \mathcal{P}^*$ can be represented as $f(x) = e^{-\alpha_2 x^2} f_1(x)$, where where $\alpha_2 \geq 0$ and $f_1(x)$ is an entire function of genus at most one. Let $r > 0$. By [41, (3.2)], we know that for $|z| < r$,

$$\overline{\lim}_{n\to\infty} \sqrt[n]{|a_n f^{(n)}(z)|} |(4|\alpha_1 \alpha_2|)^{1/2} = c < 1,$$

and consequently, there exists a positive integer m_0 such that, for $|z| \leq r$ and some constant $K > 0$,

$$|a_k||f^{(k)}(z)| \leq Kc^k \quad (k \geq m_0, 0 \leq c < 1). \tag{i}$$

Hence, there is a positive integer $m_1 > m_0$ such that, for all $m \geq m_1$,

$$K \sum_{k=m+1}^{\infty} c^k = K \frac{c^{m+1}}{1-c} < \frac{\epsilon}{3}. \tag{ii}$$

Also, there is a positive integer $N \geq m_1$ such that for $|z| \leq r$ and $v > N$,

$$\left| \sum_{k=2}^{m_1} \left(1 - \frac{1}{v}\right) \cdots \left(1 - \frac{k-1}{v}\right) a_k f^{(k)}(z) - \sum_{k=2}^{m_1} a_k f^{(k)}(z) \right| < \frac{\epsilon}{3}. \tag{iii}$$

Now, by Theorem 2.2 and Theorem 2.3, the Jensen polynomials, $g_n(\frac{x}{n})$, associated with $\varphi(x)$, are hyperbolic and converge uniformly to $\varphi(x)$ on compact subsets of $\mathbb{C}$. Since they are hyperbolic, by Theorem 3.9 we know that

$$Z_c\left(g_n\left(\frac{D}{n}\right)[f(x)]\right) \leq Z_c(f(x)),$$

and that $g_n(\frac{D}{n})[f(x)] \in \mathcal{L} - \mathcal{P}^*$. Therefore, for $v > N$ and $|z| < r$, we have, by (i), (ii), and (iii),

$$\begin{aligned}
&\left| g_v\left(\frac{D}{v}\right)[f(z)] - \varphi(D)[f(z)]\right| \\
&= |\sum_{k=2}^{m_1}\left(1-\frac{1}{v}\right)\cdots\left(1-\frac{k-1}{v}\right)a_k f^{(k)}(z) - \sum_{k=2}^{m_1} a_k f^{(k)}(z) \\
&+ \sum_{k=m_1+1}^{v}\left(1-\frac{1}{v}\right)\cdots\left(1-\frac{k-1}{v}\right)a_k f^{(k)}(z) - \sum_{k=m_1+1}^{\infty} a_k f^{(k)}(z)| \\
&< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} g_n\left(\frac{D}{n}\right)[f(x)] = \varphi(D)[f(x)]$$

uniformly on compact subsets of $\mathbb{C}$, and hence we have our desired result by Hurwitz's Theorem. □

We end this section by noting that S. Fisk, in his recent aforementioned tome [64], obtained a number of important theorems which provide sufficient conditions for a linear operator (transformation) to be a linear combination of a polynomial and its derivative, which go far beyond the results we have shown here. We cite the following theorem.

Theorem 3.11 ([64, Theorems 6.1–6.3])**.** *Let $T : \pi(\Omega) \to \pi(\Omega)$ be a linear operator (cf. Notation 1.2).*

(i) Let $\Omega = [0,\infty)$. If for all polynomials $f \in \pi(\Omega)$, the zeros of f and $T[f]$ weakly interlace (cf. Definition 3.1) and $\deg f = \deg T[f]+1$, then T is a (non-zero) scalar multiple of the derivative.

(ii) If we instead assume that $\deg f = \deg T[f]$, then there are constants a, b, c such that $T[f(x)] = af(x) + (bx+c)f'(x)$.

(iii) If for all polynomials $f \in \pi(\mathbb{R})$, the zeros of f and $T[f]$ weakly interlace and $\deg f = \deg T[f] - 1$, then there are constants a, b, c, where a and c have the same sign, such that $T[f(x)] = (b + ax)f(x) + cf'(x)$.

Proof. See the Appendix, Theorem 8.11. □

Theorem 3.11 in fact completely characterizes the linear operators T in Corollary 3.7.

4. Extensions of a Theorem of Laguerre

In this section we will prove the theorem of Laguerre (see Theorem 1.35) that was stated in Section 1 (see Theorem 4.3, Theorem 4.5 and Corollary 4.6). In addition, we extend these results to the functions in the $\mathcal{L} - \mathcal{P}^*$ class (see Corollary 4.8 and Theorem 4.10). We begin with a result that will be used to prove Theorem 1.35(i).

Proposition 4.1. *Let $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$, where $\deg f = n \geq 1$. If $\alpha \in (-\infty, -n) \cup (0, \infty)$ and β is any real number, then*

$$Z_c(T[f](x)) = Z_c(\alpha f(x) + (x+\beta)f'(x)) \leq Z_c(f(x)), \tag{4.1}$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

Proof. To prove (4.1), we begin by proving it for the case when the real zeros of f are all simple. Let $a_1 < a_2 < \ldots < a_m$ be the real zeros of f. By Rolle's Theorem, there are zeros of f', $b_1, b_2, \ldots, b_{m-1}$, such that $b_i \in (a_i, a_{i+1})$ for $i = 1, 2, \ldots, m-1$. Since the sign of $f(x)$ alternates on the intervals (a_i, a_{i+1}), we know that $f(b_i)f(b_{i+1}) < 0$ for each i. Thus,

$$T[f](b_i)T[f](b_{i+1}) = \alpha^2 f(b_i)f(b_{i+1}) < 0 \quad (i = 1, 2, \ldots, m-2), \tag{4.2}$$

and it follows that $T[f]$ has at least one real zero in each interval (b_i, b_{i+1}), for a total of at least $m-2$ real zeros in the interval (b_1, b_{m-1}). To complete the proof, we will find one more real zero of $T[f]$ on the unbounded interval (b_{m-1}, ∞). To this end, we assume, without loss of generality, that the leading coefficient a_n of f is positive. The leading coefficient of $T[f]$ then becomes $(\alpha + n)c_n$, which is of the same sign as α. Thus, as $x \to +\infty$, $f(x) \to +\infty$ and $T[f(x)] \to \text{sgn}(\alpha)\infty$. Since f has only one real zero greater than b_{m-1}, we know that $f(b_{m-1}) < 0$. Consequently, $\text{sgn}(T[f(b_{m-1})]) = \text{sgn}(\alpha f(b_{m-1})) = -\text{sgn}(\alpha)$. We then conclude that $T[f]$ has a zero in the interval (b_{m-1}, ∞). Thus, $T[f]$ has at least $m-1$ real zeros, and because $\deg T[f] = \deg f$,

$$Z_c(T[f(x)]) \leq Z_c(f(x)) + 1.$$

But the number of non-real zeros of a real polynomial must be even, and hence, $Z_c(T[f]) \leq Z_c(f)$. We have proved (4.1) in the case when the zeros of f are all simple.

In general, suppose that $f(x)$ has m real zeros, $x_1 \leq x_2 \leq \ldots \leq x_m$, which are not necessarily distinct, and $2N$ non-real zeros, $\{\nu_k \pm i\mu_k\}_{k=1}^{N}$. Then, for each $\epsilon > 0$, we define the polynomial

$$f_\epsilon(x) = a_n \left[\prod_{k=1}^{m}(x - (x_k + k\epsilon))\right]\left[\prod_{k=1}^{N}((x-\nu_k)^2 + \mu_k^2)\right].$$

For ϵ sufficiently small, the real zeros of $f_\epsilon(x)$ are all simple, and thus $Z_c(T[f_\epsilon](x)) \leq Z_c(f_\epsilon(x)) = 2N$. Also, $f_\epsilon \to f$ uniformly on compact subsets of $\mathbb{C}$ (cf. Appendix, Proposition 8.7). Thus, the sequence of functions $\{T[f_{\frac{1}{k}}](x)\}_{k=1}^{\infty}$ converges uniformly to $T[f](x)$ on compact subsets of $\mathbb{C}$ (cf. Theorem 2.5), and by Hurwitz's Theorem, $Z_c(T[f](x)) \leq 2N = Z_c(f(x))$.

□

We remark that if in Proposition 4.1, $\alpha = 0$, then inequality (4.1) is an immediate consequence of Rolle's theorem. If $\alpha = -n$, where $n = \deg f$, inequality (4.1) also holds. Indeed, suppose, for the sake of argument, that

$$Z_c(-nf(x) + (x+\beta)f'(x)) > Z_c(f(x)).$$

Then by Hurwitz's theorem, for any $\epsilon > 0$ sufficiently small, $Z_c(-(n+\epsilon)f(x) + (x+\beta)f'(x)) > Z_c(f(x))$. But this contradicts Proposition 4.1 and we conclude that (4.1) remains valid when $\alpha = -n$. Thus, we have obtained the following slight extension of the above proposition.

Proposition 4.2 (cf. [111, p.7] or [127, V # 66])**.** *Let $f(x) \in \mathbb{R}[x]$, where $\deg f = n \geq 1$. If $\alpha \in (-1, -n] \cup [0, \infty)$ and β is any real number, then*

$$Z_c(\alpha f(x) + (x+\beta)f'(x)) \leq Z_c(f(x)),$$

where $Z_c(f)$ denotes the number of non-real zeros $f(x)$, counting multiplicities. Moreover, when $\alpha \geq 0$, the linear operator $T = \alpha + xD$ is a complex zero decreasing operator.

To prove Theorem 1.35(i), we will apply Proposition 4.2 in the case when $\beta = 0$.

Theorem 4.3 (Theorem 1.35(i))**.** *Let $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$, where $n \geq 1$. Let $h(x)$ be a hyperbolic polynomial, none of whose zeros lie in the interval $(0, n)$. Then,*

$$Z_c\left(\sum_{k=0}^{n} h(k) a_k x^k\right) \leq Z_c(f(x)), \tag{4.3}$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

Proof. Since h is hyperbolic with no zeros in $(0, n)$, we may express it as the product $h(x) = c\prod_{k=1}^{m}(x + \alpha_k)$, where all of the α_k lie in $(-\infty, -n] \cup [0, \infty)$. Let $f_0(x) = \sum_{k=1}^{n} a_k x^k \in \mathbb{R}_n[x]$. Observe the following construction.

$$f_1(x) = \alpha_1 f_0(x) + x f_0'(x) = \sum_{k=1}^{n} (\alpha_1 + k) a_k x^k,$$

$$f_2(x) = \alpha_2 f_1(x) + x f_1'(x) = \sum_{k=1}^{n} (\alpha_2 + k)(\alpha_1 + k) a_k x^k,$$

$$\ldots f_m(x) = \alpha_m f_{m-1}(x) + x f_{m-1}'(x) = \sum_{k=1}^{n} (\alpha_m + k) \cdots (\alpha_1 + k) a_k x^k.$$

Note that c times the last term is precisely $\sum_{k=1}^{n} h(k) a_k x^k$. Since each of the f_i constructed above are elements of $\mathbb{R}_n[x]$, by Proposition 4.2, $Z_c(f_0) \geq Z_c(f_1) \geq \ldots \geq Z_c(f_m) = Z_c(\sum_{k=0}^{n} h(k) a_k x^k)$. $\square$

Example 4.4. In Laguerre's Theorem (Theorem 4.3), it was assumed that none of the zeros of the hyperbolic polynomial, $h(x)$, lie in the interval $(0, n)$. In general, this hypothesis is necessary, as the following example shows. Let $f(x) = (x+2)(x-4) = x^2 - 2x - 8$ and let $h(x) = x - 1$, which has a zero in the interval $(0, 2)$. Then we see that $(h(2))x^2 - 2h(1)x - h(0)8 = x^2 + 8$, which has two non-real zeros.

We next prove the second part of Theorem 1.35.

Theorem 4.5 (Theorem 1.35(ii))**.** *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be an arbitrary real polynomial of degree n. Let $\varphi(x) \in \mathcal{L} - \mathcal{P}$, and suppose that none of the zeros of φ lie in the interval $(0, n)$. Then $Z_c(\sum_{k=0}^{n} \varphi(k) a_k x^k) \leq Z_c(f(x))$.*

Proof. Let $g_n(x)$ be the Jensen polynomials associated with φ. Since $g_n \to \varphi$ uniformly on compact subsets of $\mathbb{C}$ (cf. Theorem 2.2), by Hurwitz's Theorem, there is an $N \in \mathbb{N}$ such that, for all $i \geq N$, $g_i(x)$ is hyperbolic with no zeros in the interval $(0, n)$. Thus, by Theorem 4.3,

$$Z_c\left(\sum_{k=0}^{n} g_i(k) a_k x^k\right) \leq Z_c\left(\sum_{k=0}^{n} a_k x^k\right)$$

for all $i \geq N$. Since $g_i \to \varphi$ uniformly on compact subsets of $\mathbb{C}$, $\sum_{k=0}^{n} a_k g_i(k) x^k \to \sum_{k=0}^{n} a_k \varphi(k) x^k$ uniformly on compact subsets of $\mathbb{C}$ (cf. Appendix, Proposition 8.7). Hence, by Hurwitz's Theorem, for i sufficiently large,

$$Z_c\left(\sum_{k=0}^{n} \varphi(k) a_k x^k\right) \leq Z_c\left(\sum_{k=0}^{n} g_i(k) a_k x^k\right) \leq Z_c(f(x)).$$

□

The third part of Theorem 1.35 follows as a corollary.

Corollary 4.6 (Theorem 1.35(iii))**.** *Let $\varphi \in \mathcal{L}-\mathcal{P}(-\infty, 0]$. Then the sequence $\{\varphi(k)\}_{k=1}^{\infty}$ is a complex zero decreasing sequence (cf. Definition 1.29).*

Proof. Let $T = \{\varphi(k)\}_{k=0}^{\infty}$. Let $f(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then, since φ has no zeros in the interval $(0, n)$, by Theorem 4.5,

$$Z_c(T[f(x)]) = Z_c\left(\sum_{k=0}^{n} \varphi(k) a_k x^k\right) \leq Z_c(f(x)).$$

Since $f(x)$ was chosen to be an arbitrary real polynomial, it follows that $\{\varphi(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence.

□

Remark 4.7. In order to clarify the above terminology in Corollary 4.6, we remark (cf. [46, p. 140]) that if $\varphi \in \mathcal{L}-\mathcal{P}I$, then $\varphi \in \mathcal{L}-\mathcal{P}(-\infty, 0]$ or $\varphi \in \mathcal{L}-\mathcal{P}[0, \infty)$, but that an entire function in $\mathcal{L}-\mathcal{P}(-\infty, 0]$ need not belong to $\mathcal{L}-\mathcal{P}I$. Indeed, if $\varphi(x) = \frac{1}{\Gamma(x)}$, where $\Gamma(x)$ denotes the gamma function, then $\varphi(x) \in \mathcal{L}-\mathcal{P}(-\infty, 0]$, but $\varphi(x) \notin \mathcal{L}-\mathcal{P}I$. This can be seen, for example, by looking at the Taylor coefficients of $\varphi(x) = \frac{1}{\Gamma(x)}$.

We now extend Laguerre's Theorem to the $\mathcal{L}-\mathcal{P}^*$ class of entire functions.

Corollary 4.8. *Let $f \in \mathcal{L}-\mathcal{P}^*$; that is, $f(x) = p(x)\varphi(x)$, where $p(x) \in \mathbb{R}[x]$ and $\varphi(x) \in \mathcal{L}-\mathcal{P}$. If $\alpha \geq 0$, then*

$$Z_c(\alpha f(x) + x f'(x)) \leq Z_c(f(x)), \tag{4.4}$$

where $Z_c(f)$ denotes the number of non-real zeros $f(x)$, counting multiplicities.

Proof. Since $\alpha \geq 0$, by Proposition 4.2, $T = \alpha + xD$ is a complex zero decreasing operator. We can then apply Theorem 2.8 to conclude that, for any $f \in \mathcal{L}-\mathcal{P}^*$,

$$Z_c(T[f](x)) = Z_c(\alpha f(x) + x f'(x)) \leq Z_c(f(x)).$$

□

Example 4.9. The requirement that $\alpha \geq 0$ in Corollary 4.8 is necessary, as the following example shows. Let $T = -2 + xD$, and let $f(x) = e^{-x^2}$, which is in the Laguerre-Pólya class and has no non-real zeros. Then

$$T[f](x) = -2e^{-x^2} - 2x^2e^{-x^2} = -2(x^2+1)(e^{-x^2}),$$

which has two non-real zeros.

We next extend Corollary 4.6 to the functions of the $\mathcal{L}-\mathcal{P}^*$ class of transcendental entire functions (cf. Definition 1.21).

Theorem 4.10. *Let $\varphi(x) \in \mathcal{L}-\mathcal{P}(-\infty, 0]$. If $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{L}-\mathcal{P}^*$, then*

$$Z_c\left(\sum_{k=0}^{\infty} \varphi(k) a_k x^k\right) \leq Z_c(f(x)).$$

Proof. Let $f(x) = p(x)\psi(x)$, where $p(x) \in \mathbb{R}[x]$ with $2N$ non-real zeros and $\psi(x) \in \mathcal{L}-\mathcal{P}$. Let $g_n(x)$ be the Jensen polynomials of $\psi(x)$, which are hyperbolic and approach ψ uniformly on compact subsets of $\mathbb{C}$ (cf. Theorems 2.2 and 2.3). Since they are hyperbolic, $Z_c(p(x)g_n(x)) = Z_c(f(x))$ for all n.

By Corollary 4.6, the sequence $T = \{\varphi(k)\}_{k=0}^{\infty}$ is a complex zero decreasing sequence. Thus it is a multiplier sequence, and $T[f(x)]$ is an entire function (cf. Theorem 1.28). Also,

$$Z_c\left(T[p(x)g_n(x)]\right) \leq Z_c(p(x)g_n(x)).$$

Since $T[p(x)g_n(x)] \to T[f(x)]$ uniformly on compact subsets of $\mathbb{C}$ (cf. Theorem 2.5), by Hurwitz's Theorem, for n sufficiently large (see the Appendix, Corollary 8.6),

$$Z_c\left(\sum_{k=0}^{\infty} \varphi(k) a_k x^k\right) \leq Z_c(T[p(x)g_n(x)]).$$

□

5. Special Classes of Linear Operators

It is known that any linear operator $T : \mathbb{C}[z] \to \mathbb{C}[z]$ can be represented as a formal differential operator with complex polynomial coefficients $Q_k(z)$ (see Theorem 1.14). In Theorem 5.2, we refine the results of Djokovic ([60]), and answer a question of Raitchinov ([126]). The determination of linear transformations by means of the characterization of these complex polynomials, $Q_k(z)$, is an open problem of interest (see Problem 1.18 and Problem 1.19). In this section, we will investigate the cases when (i) the polynomials $Q_k(z)$ are constant, and (ii) $Q_k(z) = b_k z^k$ ($b_k \in \mathbb{C}, k = 0, 1, 2, \ldots$). Moreover, we will establish our main results pertaining to complex zero decreasing operators (see Theorem 5.14, Theorem 5.20 and Theorem 5.23). In particular, Theorem 5.23 turns out to extend Laguerre's Theorem (Theorem 4.3), the Hermite-Poulain Theorem (Theorem 1.12) and a result of Bleecker-Csordas ([18]) (see also Remarks 5.19 and 5.23).

5.1. **The Operator $\sum_{k=0}^{\infty} Q_k(x)D^k$, when $Q_k(x)$ is Constant.** In the case when the complex polynomials $Q_k(z)$ in Theorem 1.14 are characterized to be constant, the resulting linear operator has been studied by a number of authors (see [41] and the references contained therein). In 1963, I. Raitchinov ([126]) posed the following question. Is the translation invariance (cf. Definition 5.1) of T a necessary and sufficient condition for the polynomials $Q_k(z)$ associated with T to be constant?

Definition 5.1. Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator. Let $f(z) \in \mathbb{C}[z]$ and let $T[f](z) = g(z)$. Then T is said to be *translation invariant* if $T[f(z+a)] = g(z+a)$ for all $a \in \mathbb{C}$.

In the same year, D. Djokovic([60]) proved that translation invariance was a sufficient condition for the $Q_k(z)$ to be constant for all k (Necessity can be readily established). In Theorem 5.2, we shall answer Raitchinov's question (cf. [126]) with a new proof, using a technique simpler than the one used by Djokovic.

Theorem 5.2. *Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator. The following are equivalent.*

(1) T commutes with differentiation D ($D = d/dz$).

(2) T is of the form $T = \sum_{k=0}^{\infty} c_k D^k \quad (c_k \in \mathbb{C}, k = 0, 1, 2, \ldots)$.

(3) T is translation invariant.

Proof. ((1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2)) We argue by contradiction. By Theorem 1.14, $T = \sum_{k=0}^{\infty} Q_k(z)D^k$ for some complex polynomials $Q_k(z)$. Suppose that there exists a non-constant $Q_k(z)$. Then there must be a least integer $N \in \mathbb{N}$ such that $Q_N(z)$ is non-constant and for all $n < N$, $Q_n(z) = c_n$ for some constant $c_n \in \mathbb{C}$. Let $f(z) = z^N/N!$.

Now we assume (1), and apply the operators TD and DT to $f(z)$.

$$\begin{aligned}
TD\left[\frac{z^N}{N!}\right] &= T\left[\frac{z^{N-1}}{(N-1)!}\right] = \sum_{k=0}^{N-1} c_k D^k \left[\frac{z^{N-1}}{(N-1)!}\right], \text{ and} \\
DT\left[\frac{z^N}{N!}\right] &= D\left[Q_N(z)D^N\left[\frac{z^N}{N!}\right]\right] + \sum_{k=0}^{N-1} c_k D^k\left[\frac{z^N}{N!}\right] \\
&= Q_N'(z) + \sum_{k=0}^{N-1} c_k D^k \left[\frac{z^{N-1}}{(N-1)!}\right].
\end{aligned}$$

Since $TD = DT$, $Q_N'(z) = 0$, and this is only possible if $Q_N(z)$ is constant. Thus, there is a contradiction, and (1) $\Rightarrow$ (2).

If we assume (3), then for $a \neq 0$, we observe that

$$\begin{aligned}
T[f](z+a) &= Q_N(z+a) + \sum_{k=0}^{N-1} c_k f^{(k)}(z+a) \\
&= Q_N(z+a) + \sum_{k=0}^{N-1} c_k \frac{(z+a)^{N-k}}{(N-k)!}, \text{ and} \\
T[f(z+a)] &= T\left[\frac{(z+a)^N}{N!}\right] = Q_N(z) + \sum_{k=0}^{N-1} c_k D\left[\frac{(z+a)^N}{N!}\right] \\
&= Q_N(z) + \sum_{k=0}^{N-1} c_k \frac{(z+a)^{N-k}}{(N-k)!}.
\end{aligned}$$

By (3), T is translation invariant, and thus $T[f](z+a) = T[f(z+a)]$. Hence, $Q_N(z+a) = Q_N(z)$. Since Q_N is non-constant, there is a $z_0 \in \mathbb{C}$ such that z_0 is a zero of $Q_N(z)$. Then $z_0 + a$ is also a zero of $Q_N(z)$, and in fact, for all $n \in \mathbb{N}$, $z_0 + na$ is a zero of $Q_N(z)$. Thus, $Q_N(z)$ has an infinite number of zeros, which contradicts the Fundamental Theorem of Algebra. Hence, we have a contradiction, and (3) $\Rightarrow$ (2).

((2) $\Rightarrow$ (1) and (3)) By (2), $T = \sum_{k=0}^{\infty} c_k D^k$, where $c_k \in \mathbb{C}$. We observe that, for $f(z) \in \mathbb{C}[z]$, $T[f(z)] = \sum_{k=0}^{\deg f} c_k f^{(k)}(z)$. Thus,

$$\begin{aligned}
DT[f(z)] &= D\left[\sum_{k=0}^{deg f} c_k D^k(f(z))\right] = \sum_{k=0}^{\deg f-1} c_k D^{k+1}(f(z)) \\
&= \left(\sum_{k=0}^{\deg f'} c_k D^k\right)[f'(z)] = TD[f(z)],
\end{aligned}$$

and hence, (1) follows.

Now, to prove translation invariance, let $f(z) = \sum_{j=0}^{n} a_j z^j$, and set $A_{k,j} = \frac{j!}{(j-k)!} = \frac{D^k[z^j]}{z^{j-k}}$ when $k \le j$, and $A_{k,j} = 0$ otherwise. Then a calculation shows that, for $a \in \mathbb{C}$,

$$\begin{aligned}
T[f](z+a) &= \sum_{k=0}^{\infty} c_k f^k(z+a) = \sum_{k=0}^{n} c_k f^{(k)}(z+a) \\
&= \sum_{k=0}^{n} c_k \left(\sum_{j=0}^{n} a_j A_{k,j} (z+a)^{j-k}\right), \text{ and} \\
T[f(z+a)] &= T\left[\sum_{j=0}^{n} a_j (z+a)^j\right] = \sum_{k=0}^{n} c_k \left(\sum_{j=0}^{n} a_j D^k[(z+a)^j]\right) \\
&= \sum_{k=0}^{n} c_k \left(\sum_{j=0}^{n} a_j A_{k,j} (z+a)^{j-k}\right).
\end{aligned}$$

Consequently, T is translation invariant. $\square$

Remark 5.3. It is possible to state a stronger version of Theorem 5.2. Note that in proving the implication (3) $\Rightarrow$ (2) of Theorem 5.2, we only required that $T[f](z+a) = T[f(z+a)]$ for a single $a \neq 0$.

We now provide an illustration of Theorem 5.2.

Example 5.4. Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator. We define the *Pincherle derivative* of T as the linear operator

$$T' = Tz - zT.$$

It is a known property of the Pincherle derivative that, if T is translation invariant, then T' is translation invariant. We demonstrate this below. Let T be translation invariant and $f(z) \in \mathbb{C}[z]$. For $a \in \mathbb{C}$,

$$\begin{aligned} T'[f](z+a) &= T[zf](z+a) - (z+a)T[f](z+a) \\ &= T[(z+a)f(z+a)] - (z+a)T[f(z+a)] \\ &= T[zf(z+a)] + aT[f(z+a)] - zT[f(z+a)] - aT[f(z+a)] \\ &= T[zf(z+a)] - zT[f(z+a)] = T'[f(z+a)]. \end{aligned}$$

By Theorem 5.2, the translation invariance of T is equivalent to T being of the form $T = \sum_{k=0}^{\infty} c_k D^k$ for constants c_k. It follows that if T is of the form $T = \sum_{k=0}^{\infty} c_k D^k$ for constants c_k, then T' is of the form $T' = \sum_{k=0}^{\infty} b_k D^k$ for constants b_k. Indeed, we observe that when $T = D^n$,

$$(D^n)'[f(z)] = D^n[zf(z)] - zD^n[f(z)] = (z+n)f^{(n)}(z) - zf^{(n)}(z) = nD^n[f(z)].$$

Thus, if $T = \sum_{k=0}^{\infty} c_k D^k$ for constants c_k, then $T' = \sum_{k=0}^{\infty} kc_k D^k$, which has all constant terms. Moreover, we have also shown that for $h(y) \in \mathbb{R}[y]$ hyperbolic, the Pincherle derivative of the linear operator $h(D)$ (cf. (1.1)) is the linear operator $h'(D)$.

We recall that by the Hermite-Poulain Theorem (Theorem 1.12), the condition that the polynomial $h(y) \in \mathbb{R}[y]$ is hyperbolic is a sufficient condition for the differential operator $h(D) : \mathbb{R}[x] \to \mathbb{R}[x]$ (cf. (1.1)) to be a complex zero decreasing operator. The question arises if this is also a necessary condition, for characterizing all complex zero decreasing operators whose associated polynomials $Q_k(x)$ are all constant. Indeed, the above condition is necessary, which the following theorem shows.

Theorem 5.5. *Let* $p(y) := \sum_{k=0}^{n} a_k y^k \in \mathbb{R}[y]$ *and let* $D = d/dx$. *Then* $p(D)$ *is a complex zero decreasing operator if and only if* $p(y) \in \mathcal{L} - \mathcal{P}$.

Proof. One direction is clear, for if $p(y) \in \mathcal{L} - \mathcal{P}$, then by the Hermite-Poulain Theorem (Theorem 1.12), $p(D)$ is a complex zero decreasing operator.

Conversely, suppose that $T = p(D)$ is a complex zero decreasing operator. If $p(y) = y^n$, then certainly $p(y) \in \mathcal{L} - \mathcal{P}$, and thus we assume that $p(y)$ is not a monomial. Let $m \in \mathbb{N}, m > n$. Then,

$$\begin{aligned} T[x^m] = p(D)x^m &= \sum_{k=0}^{n} a_k m(m-1)\cdots(m-k+1)x^{m-k} \\ &= x^{m-n}\sum_{k=0}^{n} a_k m(m-1)\cdots(m-k+1)x^{n-k} \end{aligned}$$

has only real zeros, as x^m had only real zeros. Therefore, $h_m(x) = \sum_{k=0}^{n} a_k m(m-1)\cdots(m-k+1)x^{n-k}$ has only real zeros. Let

$$h_m(\frac{x}{m}) = \sum_{k=0}^{n} a_k \frac{m}{m}(1-\frac{1}{m})\cdots(1-\frac{k-1}{m})x^{n-k} \in \mathcal{L}-\mathcal{P}.$$

As $m \to \infty$, we have that

$$\lim_{m\to\infty} h_m\left(\frac{x}{m}\right) = \sum_{k=0}^{n} a_k x^{n-k} = x^n \sum_{k=0}^{n} a_k x^{-k} = x^n p\left(\frac{1}{x}\right) = p^*(x),$$

where $p^*(x)$ is the *reverse* of $p(x)$ (cf. Definition 1.40). Note that this convergence is uniform. Hence, we infer from Hurwitz's Theorem that $p^*(y)$ has only real zeros. By the observation in Definition 1.40, we conclude that $p(y) \in \mathcal{L}-\mathcal{P}$, as desired. $\square$

5.2. **The Operator $\sum_{k=0}^{\infty} Q_k(x)D^k$, when $Q_k(x) = b_k x^k$.** We next investigate the case when the linear operator T is of the form

$$T = \sum_{k=0}^{\infty} b_k x^k D^k \quad (b_k \in \mathbb{R}). \tag{5.1}$$

Parts of the following preparatory result may be found in [124, vol. I, p. 8, # 44].

Lemma 5.6. *Let $q(x) := \sum_{k=0}^{n} b_k x^k \in \mathbb{R}[x]$. Let $\theta := xD$, where $D = \frac{d}{dx}$. Then*

(i) $\theta^n[x^k] = k^n x^k, \quad k, n \in \mathbb{N}$;

(ii) for any $f(x) \in \mathbb{R}[x], \theta(\theta-1)\cdots(\theta-k+1)[f(x)] = x^k f^{(k)}(x)$;

(iii) if $f(x) = \sum_{j=0}^{m} c_j x^j \in \mathbb{R}[x]$, then

$$q(\theta)[f(x)] = \sum_{j=0}^{m} c_j q(j) x^j.$$

Proof. (i) Proof by induction. Observe that

$$\theta^1[x^k] = xkx^{k-1} = k^1 x^k.$$

If $\theta^n[x^k] = k^n x^k$, then

$$\theta^{n+1}[x^k] = \theta[\theta^n[x^k]] = x(k^{n+1}x^{k-1}) = k^{n+1}x^k.$$

(ii) Proof by induction. Let $f(x) = \sum_{j=0}^{n} a_j x^j$. Observe that

$$\theta[f(x)] = xf'(x) = x^1 f^{(1)}(x).$$

If $\theta(\theta-1)\cdots(\theta-k+1)[f(x)] = x^k f^{(k)}(x)$, then

$$\theta(\theta-1)\cdots(\theta-k+1)(\theta-k)[f(x)] = \theta(\theta-1)\cdots(\theta-k+1)[xf'(x)-kf(x)]$$
$$= x^k((xf'(x)-kf(x))^{(k)}) = x^k(kf^{(k)}(x)+xf^{(k+1)}(x)-kf^{(k)}(x))$$
$$= x^{k+1}f^{(k+1)}(x).$$

(iii) Note that for any $\alpha \in \mathbb{C}$,

$$(\theta+\alpha)[x^j] = (j+\alpha)x^j. \tag{5.2}$$

But by the Fundamental Theorem of Algebra, $q(x) = c\prod_{k=1}^n (x+\alpha_k)$ for some $\alpha_k \in \mathbb{C}$. Thus by (5.2),

$$q(\theta)[f(x)] = \left(c\prod_{k=1}^n(\theta+\alpha_k)\right)\left[\sum_{j=0}^m c_j x^j\right] = \sum_{j=0}^m c_j\left(c\prod_{k=1}^n(\theta+\alpha_k)\right)[x^j]$$
$$= \sum_{j=0}^m c_j c\left(\prod_{k=1}^n(j+\alpha_k)\right)[x^j] = \sum_{j=0}^m c_j q(j)x^j.$$

□

Theorem 5.7. *Let $q(x) = \sum_{k=0}^n b_k x^k \in \mathbb{R}[x]$, where $b_0 \neq 0$. Then*

$$T = \sum_{k=0}^n b_k x^k D^k$$

is a complex zero decreasing operator if and only if the polynomial

$$\tilde{q}(x) := \sum_{k=0}^n b_k x(x-1)\cdots(x-k+1)$$

has only real negative zeros (i.e., $\tilde{q}(x) \in \mathcal{L}-\mathcal{P}^+$, see Definition 1.21).

Proof. Let $f(x) := \sum_{j=0}^m c_j x^j$ be an arbitrary real polynomial. Then by Lemma 5.6,

$$T[f(x)] = \sum_{k=0}^n b_k x^k f^{(k)}(x) = \sum_{j=0}^m c_j \tilde{q}(\theta)[x^j]. \tag{5.3}$$

Now, using the characterization of complex zero decreasing sequences interpolated by polynomials (see Theorem 1.36 in Section 1 or [42, Theorem 2.13] or [46]), the sequence $\{\tilde{q}(k)\}_{k=0}^\infty, \tilde{q}(0) \neq 0$, is a CZDS if and only if $\tilde{q}(x)$ has only real negative zeros. Hence, by (5.3), T is a CZDS if and only if $\tilde{q}(z)$ has only real negative zeros. □

We recall that in Proposition 1.39, we characterized linear operators T defined by sequences as formal differential operators of the form

$$T = \sum_{k=0}^\infty \frac{g_k^*(-1)}{k!} x^k D^k,$$

where $\frac{g_k^*(-1)}{k!} \in \mathbb{R}$. "Conversely", we see that any formal differential operator of the form $T = \sum_{k=0}^{\infty} b_k x^k D^k$ for some sequence $\{b_k\}_{k=0}^{\infty}$ of real numbers can be represented as a sequence.

Proposition 5.8. *Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator, $T = \sum_{k=0}^{\infty} b_k x^k D^k$. Then we can represent T as the sequence $\{\gamma_k\}_{k=0}^{\infty}$, where $\gamma_n = \sum_{k=0}^{n} k!\binom{n}{k} b_k$ and $T[x^n] = \gamma_n x^n$.*

Proof. We simply observe that

$$T[x^n] = \sum_{k=0}^{n} b_k x^k D^k[x^n] = x^n \left(\sum_{k=0}^{n} b_k \frac{n!}{(n-k)!} \right) = \left(\sum_{k=0}^{n} k! \binom{n}{k} b_k \right) x^n.$$

□

Indeed, with this representation, theorems about multiplier sequences and CZDS are equivalent to theorems about linear operators of the form in (5.1).

5.3. **The Main Result: Complex Zero Decreasing Operators.** Our next results involve the analysis of linear operators $T : \mathbb{R}[x] \to \mathbb{R}[x]$ of the form $T = \alpha x + \beta + (x^2 + cx + e)D$, where $D = d/dx$ and $\alpha, \beta, c, e \in \mathbb{R}$. Thus, in the representation of T as a formal differential operator (see Theorem 1.14), the polynomials $Q_k(x) \equiv 0$ for $k \geq 2$.

Proposition 5.9. *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator of the form $T = \alpha x + \beta + (x^2 + cx + e)D$, where $D = d/dx$ and $\alpha, \beta, c, e \in \mathbb{R}$, and $e > c^2/4$. If the real zeros of $f(x)$, $x_1 < x_2 < \ldots < x_m$, are simple, then $T[f(x_i)]T[f(x_{i+1})] < 0$ for all $i = 1, 2, \ldots, m-1$.*

Proof. Since $e > c^2/4$, $x^2 + cx + e > 0$ for all $x \in \mathbb{R}$. Also, because the real zeros of $f(x)$ are simple, we know that $f'(x_i)f'(x_{i+1}) < 0$ for $i = 1, \ldots, m-1$. It follows that

$$T[f](x_i)T[f](x_{i+1}) = (x_i^2 + cx_i + e)(x_{i+1}^2 + cx_{i+1} + e)f'(x_i)f'(x_{i+1}) < 0.$$

□

Theorem 5.10. *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator of the form $T = \alpha x + \beta + (x^2 + cx + e)D$, where $D = d/dx$ and $\alpha, \beta, c, e \in \mathbb{R}$. If $e > c^2/4$, then*

$$Z_c(T[f](x)) \leq Z_c(f(x)) + 2, \tag{5.4}$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

Proof. The inequality (5.4) is obvious when $n \leq 1$, and thus we assume that $n \geq 2$. We first prove inequality (5.4) under the assumption that the real zeros of f are simple. Let $x_1 < x_2 < \ldots < x_m$ denote the real zeros of $f(x)$, where $m \leq n = \deg f$. By Proposition 5.9, $T[f](x_i)T[f](x_{i+1}) < 0$ for all $i = 1, 2, \ldots, m-1$. Therefore, $T[f](x)$ has at least $m-1$ real zeros in the interval (x_1, x_m). Since $\deg T[f] = n+1$, the desired inequality

$$Z_c(T[f](x)) \leq (n+1) - (m-1) = n - m + 2 = Z_c(f(x)) + 2, \tag{5.5}$$

follows. To complete the proof when the polynomial $f(x)$ has multiple real zeros, we let $\deg f = n = m + 2N$ and let $x_1 \le x_2 \le \ldots \le x_m$ be the m real zeros of f, with $f(x) = c(\prod_{k=1}^{m}(x - x_k))(\prod_{j=1}^{N}(x^2 - 2a_j x + a_j^2 + b_j^2))$. For $\epsilon > 0$, we define

$$f_\epsilon(x) = c\left(\prod_{k=1}^{m}(x - (x_k + k\epsilon))\right)\left(\prod_{j=1}^{N}(x^2 - 2a_j x + a_j^2 + b_j^2)\right).$$

For ϵ sufficiently small, the real zeros of f_ϵ are simple, and $T[f_\epsilon](x)$ accordingly has at least $m-1$ real zeros in the interval $(x_1 + \epsilon, x_m + m\epsilon)$. The sequence of functions $\{f_{1/k}(x)\}_{k=1}^{\infty}$ converge uniformly to $f(x)$ on compact subsets of $\mathbb{C}$, and by Theorem 2.5, $T[f_{1/k}(x)] \to T[f(x)]$ uniformly on compact subsets of $\mathbb{C}$. By Hurwitz's Theorem, $T[f(x)]$ has at least $m-1$ real zeros in the interval (x_1, x_m), and since $\deg T[f] = \deg f + 1$,

$$Z_c(T[f(x)]) \le Z_c(f(x)) + 2,$$

follows. $\square$

Remark 5.11. We remark that the inequality (5.4) is sharp in the sense that equality is attained by some polynomials and appropriate choices of constants α, β, c, and e. Indeed, let $f(x) = x^3 + x$, and $T = x + (x^2+1)D$. Then, $T[f](x) = (x^4 + x^2) + (2x^2+1)(x^2+1) > 0$ for all $x \in \mathbb{R}$. Thus, $T[f](x)$ has four non-real zeros while $f(x)$ has two non-real zeros, and hence, $Z_c(T[f](x)) = Z_c(f(x)) + 2$ for that particular f.

In the case when the α in Theorem 5.10 is less than the negative of the degree of f, we can obtain a stronger result.

Theorem 5.12. *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial of degree $n \ge 2$. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator defined as $T = \alpha x + \beta + (x^2 + cx + e)D$, where $\alpha < -n$ and $e > \frac{c^2}{4}$. Then*

$$Z_c(T[f](x)) \le Z_c(f(x)), \tag{5.6}$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities. Also, the zeros of $T[f(x)]$ weakly interlace with the zeros of $f(x)$. (cf. Definition 3.1).

Proof. We first prove (5.6) in the case when the real zeros of f are simple. Let $x_1 < x_2 < \ldots < x_m$ be the m real zeros of $f(x)$. By Proposition 5.9, we know that $T[f(x_i)]T[f(x_{i+1})] < 0$ for each i, and it follows that there are at least $m-1$ real zeros of $T[f](x)$ in the interval (x_1, x_m). To complete the proof, we will find an additional real zero of $T[f](x)$ on each of the unbounded intervals (x_m, ∞) and $(-\infty, x_1)$.

Without loss of generality, assume that $a_n > 0$, where a_n is the leading coefficient of f. Then $\lim_{x\to\infty} f(x) = +\infty$, and thus, $\operatorname{sgn}(f'(x_m)) = -1$, as otherwise f has a zero in the interval (x_m, ∞). Similarly, since $\lim_{x\to-\infty} f(x) = (-1)^n\infty$, $\operatorname{sgn}(f'(x_1)) = (-1)^{n+1}$. It follows that $\operatorname{sgn}(T[f](x_1)) = \operatorname{sgn}((x_1^2 + cx_1 + e)f'(x_1)) = (-1)^{n+1}$ and $\operatorname{sgn}(T[f](x_m)) = \operatorname{sgn}((x_m^2 + cx_m + e)f'(x_m)) = +1$.

However, the leading coefficient of $T[f]$ is $(\alpha+n)a_n < 0$, and thus, $\lim_{x\to\infty} T[f](x) = -\infty$ and $\lim_{x\to-\infty} T[f](x) = (-1)^n\infty$. Hence, $T[f]$ has a zero in each of the intervals $(-\infty, x_1)$ and (x_m, ∞). This gives us a total of at least m real zeros of $T[f](x)$, and since $\deg T[f] = \deg f + 1$,

$$Z_c(T[f](x)) \leq Z_c(f(x)) + 1.$$

Since the number of non-real zeros of a real polynomial must be even, we conclude that $Z_c(T[f](x)) \leq Z_c(f(x))$. If, in addition, $f(x)$ is hyperbolic, we have also proved that the zeros of $T[f]$ strictly interlace with the zeros of f.

For the case when f has multiple zeros, we use a routine continuity argument in conjunction with Hurwitz's Theorem (see the Appendix, Theorem 8.5 and Corollary 8.8) to obtain our desired result. □

Remark 5.13. We note that the linear operator $T = \alpha x + \beta + (x^2 + cx + e)D$ in Theorem 5.12 does *not* possess the weak interlacing property (cf. Definition 3.2). This is because the zeros of $T[f(x)]$ interlace with the zeros of $f(x)$ only if $\deg f < -\alpha$. Thus, Theorem 5.12 does not contradict Theorem 3.11(iii). Indeed, the following concrete example shows that if $\deg f > -\alpha$, then (5.6) does not hold. If $T = -3x + (x^2 + 1)D$ and $f(x) = x^2(x^2 - 1)$, then $T[f(x)] = x(x^4 + 5x^2 - 2)$, which has non-real zeros.

The following result, which appears to be new, turns out to generalize many of our previous results.

Theorem 5.14. *Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a real polynomial of degree $n \geq 2$ and let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator defined as $T = \alpha x + (\delta x^2 + cx + e)D$, where $\delta \geq 0$, $c, e \in \mathbb{R}$, α is outside of the interval $[-\delta n, 0]$ and $\alpha \cdot e < 0$. Then,*

$$Z_c(T[f](x)) \leq Z_c(f(x)), \tag{5.7}$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

Proof. We first prove (5.7) under the additional assumption that the real zeros of f are simple. We also make use of a theorem on the perturbation of zeros (see the Appendix, Corollary 8.8) and thus we may assume, without loss of generality, that f' does not have a zero at 0.

Let $a_1 < a_2 < \ldots < a_m$ be the m real zeros of f. We know by Rolle's Theorem that there are an odd number of zeros of f' in each interval (a_i, a_{i+1}). We can then choose $b_1, b_2, \ldots, b_{m-1}$ to be zeros of f' selected so that b_1 is the smallest zero of f' in (a_1, a_2), and b_i is the largest zero of f' in (a_i, a_{i+1}) for $i = 2, 3, \ldots, m-2$. We know then that $f(b_i)f''(b_i) < 0$ for each i. Since the zeros of f are simple, the sign of f alternates on the intervals (a_i, a_{i+1}), and it follows that $f(b_i)f(b_{i+1}) < 0$ $(i = 1, \ldots, m-2)$. Consequently, whenever $b_i b_{i+1} > 0$,

$$T[f](b_i)T[f](b_{i+1}) = \alpha^2 b_i b_{i+1} f(b_i) f(b_{i+1}) < 0. \tag{5.8}$$

Since $b_i b_{i+1} > 0$ for all but at most one pair b_i, b_{i+1}, we have found at least $m-3$ real zeros of $T[f](x)$ in the interval (b_1, b_{m-1}). To complete the proof, we must find the remaining zeros of the $(m+1)-$st degree polynomial $T[f](x)$ in the unbounded intervals (b_{m-1}, ∞) and $(-\infty, b_1)$ as well as in the interval (b_i, b_{i+1}), if there is a b_i such that $b_i b_{i+1} < 0$.

We assume, without loss of generality, that $c_n > 0$, and note that the leading coefficient of x^{n+1} in the $(n+1)^{st}$ degree polynomial $T[f](x)$ is $(\alpha + \delta n)c_n$, which is of the same sign as α since $\alpha < -\delta n$ or $\alpha > 0$. Since $\lim_{x\to\infty} f(x) = +\infty$, we know that $\lim_{x\to\infty} T[f(x)] = \text{sgn}(\alpha)\infty$, $\text{sgn}f = +1$ on the interval (a_m, ∞), and thus, $\text{sgn}(f(b_{m-1})) = -1$. Similarly, since $\lim_{x\to-\infty} f(x) = (-1)^n\infty$, we know that $\lim_{x\to-\infty} T[f(x)] = (-1)^{n+1}\text{sgn}(\alpha)\infty$, $\text{sgn}f = (-1)^n$ on the interval $(-\infty, a_1)$, and thus, $\text{sgn}(f(b_1)) = (-1)^{n+1}$. Aided by the foregoing analysis, we establish the following facts.

Fact 1. If $b_{m-1} > 0$, then $\text{sgn}(T[f](b_{m-1})) = \text{sgn}(\alpha b_{m-1} f(b_{m-1})) = -\text{sgn}(\alpha)$. Thus, $T[f]$ has at least one real zero in the interval (b_{m-1}, ∞).

Fact 2. If $b_1 < 0$, then $\text{sgn}(T[f](b_1)) = \text{sgn}(\alpha b_1 f(b_1)) = \text{sgn}(\alpha(-1)^n) = (-1)^n\text{sgn}(\alpha)$. Thus, $T[f]$ has at least one zero in the interval $(-\infty, b_1)$.

Fact 3. If $b_1 > 0$, then $\text{sgn}(T[f](b_1)) = \text{sgn}(\alpha f(b_1)) = (-1)^{n+1}\text{sgn}(\alpha)$. If f' has no zeros in the interval $(0, b_1)$, then $\text{sgn}(f'(0)) = (-1)^{n+1}$, and

$$\text{sgn}(T[f(0)]) = \text{sgn}(ef'(0)) = -\text{sgn}(\alpha f'(0)) = (-1)^n\text{sgn}(\alpha),$$

which means that $T[f]$ has at least two zeros in the interval $(-\infty, b_1)$. We now assume that f' has a greatest zero b' in the interval $(0, b_1)$. Since b_1 is the smallest zero of f' in the interval (a_1, a_2), b' is the greatest zero of f' in the interval $(0, a_1)$. Then, $\text{sgn}(f(b')) = -\text{sgn}(f(b_1))$ as $b' < a_1 < b_1$, and

$$\text{sgn}(T[f(b')]) = \text{sgn}(\alpha b' f(b')) = (-1)^n\text{sgn}(\alpha).$$

Thus, $T[f]$ has at least two zeros in the interval $(-\infty, b_1)$. We conclude that if $b_1 > 0$, then $T[f]$ has at least two zeros in $(-\infty, b_1)$.

Fact 4. If $b_{m-1} < 0$, then $\text{sgn}(T[f](b_{m-1})) = -\text{sgn}(\alpha f(b_{m-1})) = \text{sgn}(\alpha)$. If f' has no zeros in the interval $(b_{m-1}, 0)$, then $\text{sgn}(f'(0)) = +1$ and

$$\text{sgn}(T[f(0)]) = \text{sgn}(ef'(0)) = -\text{sgn}(\alpha),$$

and it follows that $T[f]$ has at least two zeros in the interval (b_{m-1}, ∞). Now, let b^* be the smallest zero of f' in the interval $(a_m, 0)$. Since b_{m-1} is the greatest zero of f' in the interval (a_{m-1}, a_m), b^* is the smallest zero of f' in the interval $(a_m, 0)$. Then $\text{sgn}(f(b^*)) = -\text{sgn}(f(b_{m-1}))$, as $b_{m-1} < a_m < b^*$, and

$$\text{sgn}(T[f(b^*)]) = \text{sgn}(\alpha b^* f(b^*)) = -\text{sgn}(\alpha).$$

Thus, $T[f]$ has at least two zeros in the interval (b_{m-1}, ∞). We conclude that if $b_{m-1} < 0$, then $T[f]$ has at least two zeros in (b_{m-1}, ∞).

Fact 5. If $b_k b_{k+1} < 0$, then let us pick $b'_{k+1} \in (a_{k+1}, a_{k+2})$ such that b'_{k+1} is the smallest zero of f' in the interval (a_{k+1}, a_{k+2}). If $b'_{k+1} > 0$, then

$$T[f](b_k)T[f](b'_{k+1}) = \alpha^2 b_k b'_{k+1} f(b_k) f(b'_{k+1}) > 0. \tag{5.9}$$

Let $g(x) = T[f](x) - ef'(x) = x(\alpha f(x) + (x + c)f'(x))$. Then

$$g(b_k)g(b'_{k+1}) = \alpha^2 b_k b'_{k+1} f(b_k) f(b'_{k+1}) > 0,$$

and thus, $\text{sgn}(g(b_k)) = \text{sgn}(g(b'_{k+1}))$. But since $g(0) = 0$, g has a zero in the interval (b_k, b'_{k+1}). Consequently, g has an even number of zeros in the interval (b_k, b'_{k+1}), and it follows that there is a point $y \in (b_k, b'_{k+1})$ such that

$$\text{sgn}(g(y)) = -\text{sgn}(g(b'_{k+1})) = -\text{sgn}(\alpha f(b'_{k+1})).$$

Since b_k and b'_{k+1} are consecutive zeros of f', the sign of $ef'(x)$ is constant on the interval (b_k, b'_{k+1}). Thus,

$$\text{sgn}(ef'(y)) = \text{sgn}(e(f(b'_{k+1}) - f(b_k))) = \text{sgn}(ef(b'_{k+1})) = -\text{sgn}\alpha f(b'_{k+1}),$$

and we get that

$$\text{sgn}(T[f](y)) = \text{sgn}(g(y) + ef'(y)) = -\text{sgn}(\alpha f(b'_{k+1})) = -\text{sgn}(T[f](b'_{k+1})).$$

By (5.9), $T[f]$ has at least two zeros in the interval $(b_k, b'_{k+1}) \subset (b_k, b_{k+1})$. If $b'_{k+1} < 0$, then

$$T[f](b_k)T[f](b'_{k+1}) = \alpha^2 b_k b'_{k+1} f(b_k) f(b'_{k+1}) < 0.$$

Also, since $b_{k+1}, b'_{k+1} \in (a_{k+1}, a_{k+2})$, $f(b'_{k+1})f(b_{k+1}) > 0$. Thus,

$$T[f](b'_{k+1})T[f](b_{k+1}) = \alpha^2 b'_{k+1} b_{k+1} f(b'_{k+1}) f(b_{k+1}) < 0.$$

Hence, $T[f]$ has at least two zeros in the interval (b_k, b_{k+1}).

Using the above facts, we can locate $m+1$ real zeros of $T[f]$, regardless of the way the zeros of f' are arranged. In order to justify this claim, we consider the following three cases.

Case 1: $b_1, b_2, \ldots, b_{m-1} > 0$. Then by (5.8), Fact 1, and Fact 3, $T[f]$ has $m-2$ zeros in the interval (b_1, b_{m-1}), one zero in the interval (b_{m-1}, ∞), and two zeros in the interval $(-\infty, b_1)$, for a total of $m+1$ real zeros.

Case 2: $b_1, b_2, \ldots, b_{m-1} < 0$. Then by (5.8), Fact 2, and Fact 4, $T[f]$ has $m-2$ zeros in the interval (b_1, b_{m-1}), one zero in the interval $(-\infty, b_1)$, and two zeros in the interval (b_{m-1}, ∞), for a total of $m+1$ real zeros.

Case 3: $b_k < 0 < b_{k+1}$ for some $k = 1, \ldots, m-2$. Then by (5.8), Fact 1, Fact 2, and Fact 5, $T[f]$ has $m-3$ zeros in the union of intervals $(b_1, b_k) \cup (b_{k+1}, b_{m-1})$, one zero in the interval $(-\infty, b_1)$, one zero in the interval (b_{m-1}, ∞), and two zeros in the interval (b_k, b_{k+1}), for a total of $m+1$ real zeros.

Thus, in all cases, $Z_c(T[f]) \leq n+1-(m+1) = Z_c(f)$. The extension to the case where the zeros of f are not simple is a routine continuity argument in conjunction with Hurwitz's Theorem (see the Appendix, Corollary 8.8). $\square$

Remark 5.15. Similar to the way Proposition 4.2 extended Proposition 4.1, Theorem 5.14 can be extended to the case when $\alpha = -\delta n$. Let $e, \delta \geq 0$ and $c \in \mathbb{R}$. If we suppose that

$$Z_c(-nxf(x) + (\delta x^2 + cx + e)f'(x)) > Z_c(f(x)),$$

then by Hurwitz's Theorem, for any $\epsilon > 0$ sufficiently small, $Z_c(-(n+\epsilon)xf(x) + (\delta x^2 + cx + e)f'(x)) > Z_c(f(x))$, which contradicts Theorem 5.14. It also can be extended to the case when $\alpha = 0$, as then we have that $T[f](x) = (\delta x^2+cx+e)f'(x)$, and it follows that $Z_c(T[f](x)) \leq Z_c(f(x))$ whenever $e \leq 0$. We can now state the following extension of Theorem 5.14.

Corollary 5.16. *Let $f(x) = \sum_{k=0}^{n} c_k x^k$ be a real polynomial of degree $n \geq 2$ and let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator defined as $T = \alpha x + (\delta x^2 + cx + e)D$, where $\delta \geq 0$, $c \in \mathbb{R}$, α is outside of the interval $(-\delta n, 0)$ and one of the conditions*

(i) $\alpha \cdot e < 0$,
(ii) $\alpha = 0, e \leq 0$,
(iii) $e = 0$

are satisfied. Then

$$Z_c(T[f](x)) \leq Z_c(f(x)),$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

It is possible to extend the result of Theorem 5.14 by taking advantage of the fact that the number and simplicity of the zeros of a polynomial are invariant under translation.

Corollary 5.17. *Let f be a real polynomial of degree $n \geq 2$. Let $T = \alpha x + \beta + (\delta x^2 + cx + e)D$ be a linear operator. If $\delta \geq 0$, $c, \beta \in \mathbb{R}$, $\alpha \notin (-\delta n, 0]$ and*

$$sgn(e + \delta(\beta/\alpha)^2 - c(\beta/\alpha)) = -sgn(\alpha), \tag{5.10}$$

then

$$Z_c(T[f](x)) \leq Z_c(f(x)),$$

where $Z_c(f(x))$ denotes the number of non-real zeros of $f(x)$, counting multiplicities.

Proof. Let $f(x)$ be a real polynomial of degree $n \geq 2$. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator, $T = \alpha x+\beta+(\delta x^2+cx+e)D$, with $\delta > 0$ and $\alpha \notin (-\delta n, 0]$. Defining $h(x) = f(x - \beta/\alpha)$, we get that

$$T[f](x-\beta/\alpha) = \alpha x h(x) + (\delta x^2 + (c - 2\delta(\beta/\alpha))x + (e + \delta(\beta/\alpha)^2 - c(\beta/\alpha)))h'(x).$$

Define the linear operator $S : \mathbb{R}[x] \to \mathbb{R}[x]$ as

$$S = \alpha x + (\delta x^2 + (c - 2\delta(\beta/\alpha))x + (e + \delta(\beta/\alpha)^2 - c(\beta/\alpha)))D,$$

and whence $S[h(x)] = T[f](x - \beta/\alpha)$. By Theorem 5.14, $Z_c(S[h(x)]) \leq Z_c(h(x))$ when $\mathrm{sgn}(e + \delta(\beta/\alpha)^2 - c(\beta/\alpha)) = -\mathrm{sgn}(\alpha)$. Since the number of non-real zeros is invariant under translation by a real constant, $Z_c(T[f](x - \beta/\alpha)) = Z_c(T[f](x))$ and $Z_c(h(x)) = Z_c(f(x))$. These facts show that, when (5.10) is satisfied,

$$Z_c(T[f](x)) = Z_c(S[h](x)) \leq Z_c(h(x)) = Z_c(f(x)).$$

□

Remark 5.18. We note that in Corollary 5.17, when $\delta = 1$ and $e \geq c^2/4$, the expression on the left-hand side of (5.10) becomes

$$e + (\beta/\alpha)^2 - c(\beta/\alpha) \geq c^2/4 + (\beta/\alpha)^2 - c(\beta/\alpha) = (\beta/\alpha - c/2)^2, \tag{5.11}$$

which is always positive regardless of the β chosen. Thus, when $\alpha < -n$, the equation in (5.10) is satisfied. This theorem is thus an extension of Theorem 5.12.

Unfortunately, we cannot get an analogous result for the case when $\alpha > 0$; that is, we cannot place a restriction on e relative to c to ensure that (5.10) is always satisfied. Observe that if $\alpha > 0$, then $e+(\beta/\alpha)^2-c(\beta/\alpha) < 0$ must be true to satisfy (5.10). However, $(\beta/\alpha)^2$ can be arbitrarily large and positive, far outweighing the other terms in the expression. Of course, we note that although $e < c^2/4$ is not a sufficient condition for (5.10) to be satisfied when $\alpha \geq 0$, it is a necessary one. Indeed, when $e \geq c^2/4$, (5.11) still holds, and thus (5.10) is false.

Remark 5.19. Indeed, Theorem 5.14 and its extensions turn out to be a generalization of many previous results. Letting

$$T = \alpha x + (\delta x^2 + cx + e)D, \tag{5.12}$$

we obtain the following.

1. If $e = 0$ and $\delta = 1$, then $T = x(\alpha + (x + c)D)$, and $Z_c(T[f(x)]) \leq Z_c(f(x))$ whenever $\alpha \notin (-\deg f, 0)$ and $c \in \mathbb{R}$. Of course,

$$Z_c(T[f(x)]) = Z_c(x((\alpha + (x + c)D)[f(x)])) = Z_c((\alpha + (x + c)D)[f(x)]).$$

Thus, Corollary 5.16 is an extension of Proposition 4.2, which itself is an extension of Laguerre's Theorem (cf. Theorem 4.3).

2. If $\delta, e = 0$, $c \in \mathbb{R}$ and $\alpha \geq 0$, then $T = x(\alpha + cD) = cx(\alpha/c + D)$ is a complex zero decreasing operator by Corollary 5.16. Since $c \in \mathbb{R}$, α/c can be any real number, and this becomes an extension of the Hermite-Poulain Theorem (cf. Theorem 1.12).

In Corollary 5.16, if $\alpha, \delta \geq 0$, $c \in \mathbb{R}$ and $e \leq 0$, then $T = \alpha x + (\delta x^2 + cx + e)D$ is a complex zero decreasing operator. We can then apply Theorem 2.8 to obtain the following extension to the $\mathcal{L} - \mathcal{P}^*$ class.

Theorem 5.20. *Let $f \in \mathcal{L} - \mathcal{P}^*$ and let $T = \alpha x + (\delta x^2 + cx + e)D$ be a linear operator. If $\alpha, \delta \geq 0$, $c \in \mathbb{R}$ and $e \leq 0$, then $Z_c(T[f(x)]) \leq Z_c(f(x))$.*

The following corollaries are consequences of Corollary 2.6 and Theorem 2.8.

Corollary 5.21. *Let $f \in \mathcal{L} - \mathcal{P}$. Let $T = \alpha x + (\delta x^2 + cx + e)D$ be a linear operator. If $\alpha, \delta \geq 0$, $c \in \mathbb{R}$ and $e \leq 0$, then $T[f] \in \mathcal{L} - \mathcal{P}$.*

Corollary 5.22. *Let $f \in \mathcal{L} - \mathcal{P}^*$. Let $T = \alpha x + (\delta x^2 + cx + e)D$ be a linear operator. If $\alpha, \delta \geq 0$, $c \in \mathbb{R}$, $e \leq 0$ and the order of $f(x)$, $\rho(f(x)) < 2$, then $T[f] \in \mathcal{L} - \mathcal{P}^*$.*

If $\varphi(x) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \in \mathcal{L} - \mathcal{P}^*$, then the entire function $T[\varphi(x)]$ can be expressed as

$$T[\varphi(x)] = \sum_{k=0}^{\infty} \frac{\alpha\gamma_{k-1}}{(k-1)!} + \frac{e\gamma_{k+1}}{(k)!} + \frac{c\gamma_k}{(k-1)!} + \frac{\gamma_{k-1}}{(k-2)!}x^k$$
$$= \sum_{k=0}^{\infty} \frac{(\alpha+k-1)k\gamma_{k-1} + ck\gamma_k + e\gamma_{k+1}}{k!}x^k.$$

The k^{th} Taylor coefficient of $T[\varphi(x)]$ is then $k(\alpha+k-1)\gamma_{k-1} + ck\gamma_k + e\gamma_{k+1}$.

It is actually possible to extend Theorem 5.14 even further by adding a D^2 term to the linear operator.

Theorem 5.23. *Let $f(x) = \sum_{k=0}^{n} c_k x^k \in \mathbb{R}[x]$ be a polynomial of degree $n \geq 2$. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator of the form $\alpha x + (\delta x^2 + cx + e)D + (rx^3 + sx)D^2$. If $c \in \mathbb{R}$, $\delta \geq 0$ and the following conditions hold,*

(i) $\alpha \notin [-\delta n, 0]$,

(ii) $\alpha \cdot e < 0$,

(iii) $r\alpha \leq 0, s\alpha \leq 0, rs \geq 0$,

(iv) $\alpha < -\delta n$, $\alpha + r(n)(n-1) < -\delta n$ *or* $\alpha > 0$, $\alpha + r(n)(n-1) > 0$, *then*

$$Z_c(T[f(x)]) \leq Z_c(f(x)).$$

Proof. We shall assume that $r, s \neq 0$, as the case when $rs = 0, r + s \neq 0$ is very similar, and the case when $r, s = 0$ is simply Theorem 5.14. First, suppose that the real zeros of f are simple. Let $a_1 < a_2 < \ldots < a_n$ be the real zeros of f. Define the linear operator $S : \mathbb{R}[x] \to \mathbb{R}[x]$ as $S := \alpha x + (x^2 + cx + e)D$, where $T = S + (rx^3 + sx)D^2$. Note that by Conditions (i), (ii), and (iv), S satisfies the conditions for the linear operator in Theorem 5.14. We now derive the following facts.

Fact 5.24. Assuming that $c_n > 0$, the leading coefficient of $T[f](x)$ is $(\alpha + \delta n + r(n)(n-1))c_n$, which is of the same sign as α by Condition (iv). Indeed, this means that the leading coefficient of $T[f](x)$ is of the same sign as the leading coefficient of $S[f](x)$.

Fact 5.25. Let $b \in \mathbb{R}$ be the greatest (or least) zero of f' in one of the intervals (a_i, a_{i+1}), or in one of the unbounded intervals $(-\infty, a_1)$ or (a_n, ∞). If $f''(b) = 0$, then $T[f(b)] = S[f(b)]$. Otherwise, we know that $\text{sgn}(f(b)) = -\text{sgn}(f''(b))$, and

$$T[f(b)] = S[f(b)] + (rx^3 + sx)D^2[f(b)] = \alpha b f(b) + (rb^2 + s)bf''(b).$$

Note that $\text{sgn}(rb^2+s) = \text{sgn}(r)$, as $\text{sgn}(r) = \text{sgn}(s)$ and $b^2 \geq 0$. Thus, $\text{sgn}(\alpha b f(b)) = \text{sgn}((rb^2 + s)bf''(b))$, consequently, $\text{sgn}(T[f(b)]) = \text{sgn}(S[f(b)])$ for all such points b.

Fact 5.26. When $T[f(x)]$ is evaluated at zero, we get

$$T[f(0)] = ef'(0) = S[f(0)].$$

The proof of Theorem 5.14 for operators S relied on the sign of the leading coefficient of $S[f]$, the sign of $S[f]$ at certain zeros of f', and the sign of $S[f]$ at 0. We have just shown that the sign of $T[f]$ is identical to the sign of $S[f]$ at all these points, and that the sign of the leading coefficients of $T[f]$ and $S[f]$ are also the same. By following the proof of Theorem 5.14, we obtain that $Z_c(T[f](x)) \leq Z_c(f(x))$. □

Remark 5.27. Theorem 5.23 is a generalization of a result by Bleecker-Csordas [18, Lemma 2.2]. If $\alpha > 0$, $\delta = 1$ and $c, e, r = 0$, then the linear operator T becomes

$$T = x(\alpha + xD + sD^2),$$

which is a complex zero decreasing operator with the condition that $s \leq 0$ and $\alpha - s > 0$.

6. Complex Zero Increasing Operators and Positivity

We begin this section with a brief treatment of the *inverse* of a linear operator (see Definition 6.1). We also define complex zero increasing operators (see Definition 6.2), and prove a theorem that states that the inverse of a complex zero decreasing operator is a complex zero increasing operator (see Proposition 6.5). We then use this theorem to provide necessary and sufficient conditions for a linear operator to be a complex zero increasing operator (see Corollary 6.6, Corollary 6.8 and Proposition 6.9). We conclude this section by establishing an integral formula for the infinite-order differential operator, e^{D^2} (see Example 6.10), and use this formula to prove that e^{D^2} preserves positivity of polynomials (see Definition 6.12 and Corollary 6.13).

Definition 6.1. Let $T : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ be a linear operator. We say that T is *invertible* if the $(n+1) \times (n+1)$ matrix , M_T, associated with T (cf. Remark 1.20) is an invertible matrix. A linear operator $T^{-1} : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ is called the *inverse* of T if its matrix representation, $M_{T^{-1}}$, satisfies $M_T M_{T^{-1}} = I = M_{T^{-1}} M_T$. Note that this is equivalent to the assertion that $TT^{-1} = I = T^{-1}T$.

We remark that $T : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ is invertible if and only if T is one-to-one.

Definition 6.2. Let $T : \mathbb{R}[x] \to \mathbb{R}[x]$ be a linear operator. We say that T is a *complex zero increasing operator* if, whenever $f(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}_n[x]$, then

$$Z_c(T[f(x)]) \geq Z_c(f(x)).$$

A special class of complex zero increasing operators is presented in the following definition.

Definition 6.3. Let $T = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. We say that T is a *complex zero increasing sequence* if, whenever $f(x) = \sum_{k=0}^{n} a_k x^k$ is a real polynomial, then

$$Z_c\left(\sum_{k=0}^{n} \gamma_k a_k x^k\right) \geq Z_c(f(x)).$$

At this point, we wish to establish some results concerning the relation between the inverse of a complex zero decreasing operators and the complex zero increasing operator. However, we must exercise caution, as inverses were only defined for linear operators acting on the finite-dimensional vector space $\mathbb{R}_n[x]$. For this reason, we introduce the following notation.

Notation 6.4. If $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is a linear operator, expressed in the form

$$T = \sum_{k=0}^{\infty} Q_k(x) D^k,$$

then the linear operator $T_n : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ is defined by

$$T_n := \sum_{k=0}^{n} Q_k(x) D^k,$$

provided that $\deg Q_k \leq n - k$ for each $k = 0, 1, 2, \ldots, n$. If

$$Z_c(T_n[f(x)]) \leq Z_c(f(x))$$

for all $f(x) \in \mathbb{R}_n[x]$, then we say that T_n (or T) is a complex zero decreasing operator on $\mathbb{R}_n[x]$. Similarly, if

$$Z_c(T_n[f(x)]) \geq Z_c(f(x))$$

for all $f(x) \in \mathbb{R}_n[x]$, then we say that T_n (or T) is a complex zero increasing operator on $\mathbb{R}_n[x]$.

Proposition 6.5. *If $T : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ is invertible, then it is a complex zero increasing operator on $\mathbb{R}_n[x]$ if and only if its inverse, T^{-1}, is a complex zero decreasing operator on $\mathbb{R}_n[x]$.*

Proof. Let $f(x) \in \mathbb{R}_n[x]$ be an arbitrary real polynomial. Suppose that T is a complex zero decreasing operator. Then, since $T^{-1}[f(x)] \in \mathbb{R}_n[x]$,

$$Z_c(T[T^{-1}[f(x)]]) \leq Z_c(T^{-1}[f(x)]),$$

and we simply note that $T[T^{-1}[f(x)]] = f(x)$. Similarly, if we assume that T^{-1} is a complex zero increasing operator, then since $T[f(x)] \in \mathbb{R}_n[x]$,

$$Z_c(T^{-1}[T[f(x)]]) \geq Z_c(T[f(x)]),$$

and with $T^{-1}[T[f(x)]] = f(x)$ we are done. $\square$

Note that an operator $T = \{\gamma_k\}_{k=0}^{\infty}$ defined by a sequence is invertible if and only if none of its terms are zero. Furthermore, the inverse is $\{\frac{1}{\gamma_k}\}_{k=0}^{\infty}$. The following corollary is immediate.

Corollary 6.6. *If $T = \{\gamma_k\}_{k=0}^{\infty}$ is a sequence with no zero terms, then T is complex zero increasing sequence if and only if the sequence $T^{-1} = \{\frac{1}{\gamma_k}\}_{k=0}^{\infty}$ is a complex zero decreasing sequence.*

However, since *non-invertible* complex zero decreasing sequences $\{\gamma_k\}_{k=0}^{\infty}$ exist (for example, $T = (1, 0, \ldots)$), Corollary 6.6 is not a complete characterization of complex zero increasing sequences.

Example 6.7. We note that if $T = \{\gamma_k\}_{k=0}^{\infty}$ is an invertible multiplier sequence, it is not necessarily true that T^{-1} is a multiplier sequence For example, the sequence $T = \{k+1\}_{k=0}^{\infty}$ is certainly a multiplier sequence, but the polynomial $T^{-1}[x^2 + 2x + 1] = x^2/3 + x + 1$ has two non-real zeros.

We recall that in Section 5.1 we discussed linear operators T which were of the form $T = \sum_{k=0}^{\infty} c_k D^k$, where the c_k are all constant. If T acts on $\mathbb{R}_n[x]$, then the matrix representation of T becomes

$$M_T = \left[c_{j-i} \cdot \frac{(j-1)!}{(i-1)!}\right]_{i,j}, \qquad \text{where } c_{j-i} = 0 \text{ when } i > j.$$

This is an upper triangular matrix, and if $c_0 \neq 0$, then it is an invertible upper triangular matrix. Now, suppose that $h(y) = \sum_{k=0}^{n} c_k x^k$ is a real polynomial, and let $g(y) = \sum_{k=0}^{\infty} b_k x^k$ be the power series such that $h(y)g(y) = 1$. Let $h(D)$ and $g(D)$ (cf. (1.1)) be linear operators on $\mathbb{R}[x]$. When the linear operator $h(D)g(D)$ is applied to the monomial x^m, a calculation shows that

$$\begin{aligned} h(D)g(D)[x^m] &= h(D)\left[\sum_{i=0}^{\infty} b_i D^i [x^m]\right] \\ &= \sum_{j=0}^{n} c_j D^j \left[\sum_{i=0}^{\infty} b_i D^i [x^m]\right] \\ &= \sum_{k=0}^{m} \left(\sum_{j=0}^{m-k} c_j D^j [b_{m-j-k} D^{m-j-k}[x^m]]\right) \\ &= \sum_{k=0}^{m} \left(\sum_{j=0}^{m-k} c_j b_{m-j-k} \frac{m!}{k!}\right) x^k. \\ &= \sum_{k=0}^{m} \frac{m!}{k!} \left(\sum_{j=0}^{m-k} c_j b_{m-j-k}\right) x^k. \end{aligned}$$

Of course, $\sum_{j=0}^{m-k} c_j b_{m-j-k}$ is precisely the $(m-k)^{th}$ coefficient of the Cauchy product of $h(y)g(y)$. Thus, it is zero unless $m - k = 0$. Therefore,

$$h(D)g(D)[x^m] = \frac{m!}{m!}(c_0 b_0)\, x^m = x^m.$$

Since m was any integer, it follows that $h(D)g(D) = 1$. This shows that $h(D)_m^{-1} = g(D)_m$ for all positive integers m. With this in mind, we establish the following corollary.

Corollary 6.8. *Let $h(y) = \sum_{k=0}^{n} c_k x^k$ be a real polynomial and let $g(y) = \sum_{k=0}^{\infty} b_k x^k$ be a power series such that $h(y)g(y) = 1$. Denote $T = g(D)$ and $S = h(D)$, where $D = d/dx$. Then $h(y)$ is hyperbolic if and only if T is a complex zero increasing operator.*

Proof. Suppose that $h(y)$ is hyperbolic. By Theorem 1.12, $h(D) = S$ is a complex zero decreasing operator on $\mathbb{R}[x]$. Thus, it is a complex zero decreasing operator on $\mathbb{R}_m[x]$ for all integers $m \geq n$. By Proposition 6.5, for all integers $m \geq n$, T is a complex zero increasing operator on $\mathbb{R}_m[x]$. Thus, whenever $f(x)$ is a real polynomial such that $\deg f \leq m$, $Z_c(T[f(x)]) = Z_c(T_m[f(x)]) \geq Z_c(f(x))$. Since this true for all m, T is a complex zero increasing operator.

Now, suppose that $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is a complex zero increasing operator. We know that $T_m = g_m(D)$, and that the inverse of $g_m(D)$ on $\mathbb{R}_m[x]$ is $h(D)$. By

Proposition 6.5, $h(D)$ is a complex zero decreasing operator on $\mathbb{R}_m[x]$. Since this is true for all $m \geq n$, it follows that $h(D)$ is a complex zero decreasing operator on $\mathbb{R}[x]$. $\square$

The quintessential complex zero decreasing operator is the differential operator $D = d/dx$. It appears to make sense that the *antiderivative* operator, the reverse of the differential operator, is a complex zero increasing operator.

Proposition 6.9. *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a real polynomial. We define the linear operator $D^{-1} : \mathbb{R}[x] \to \mathbb{R}[x]$ as*

$$D^{-1}[f(x)] = \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1}.$$

Then, D^{-1} is a complex zero increasing operator.

Proof. If $f(x) \in \mathbb{R}[x]$, then $D[D^{-1}[f(x)]] = f(x)$. Since D is a complex zero decreasing operator and $D^{-1}[f(x)]$ is a real polynomial,

$$Z_c(f(x)) = Z_c(D[D^{-1}[f(x)]]) \leq Z_c(D^{-1}[f(x)])$$

for all real polynomials $f(x)$. Thus, D^{-1} is a complex zero increasing operator. We note here that D^{-1} unfortunately does not commute with D. $\square$

Since $\varphi(x) = e^{-x^2} \in \mathcal{L} - \mathcal{P}$, we can infer from the Hermite-Poulain Theorem that $T = \varphi(D)$ is a complex zero decreasing operator. Thus, it follows from Proposition 6.5 that e^{D^2} is a complex zero increasing operator. Our interest in these types of linear operators is based, in part, on the observation that if T is a complex zero increasing operator, then $T[p](x) > 0$ for all $x \in \mathbb{R}$ whenever $p(x) \in \mathbb{R}[x]$ and $p(x) > 0$ for all $x \in \mathbb{R}$. In the following example, we present an elegant integral representation for the linear operator e^{D^2}, and demonstrate directly that whenever $f(x) > 0$ for all $x \in \mathbb{R}$, $e^{D^2}[f(x)] > 0$ for all $x \in \mathbb{R}$.

Example 6.10. Let $h(y) = e^{y^2}$ and let $D = d/dz$. Then $h(D) : \mathbb{C}[z] \to \mathbb{C}[z]$ is defined by $h(D)[f(z)] = \sum_{j=0}^{\infty} \frac{f^{(2j)}(z)}{j!}$ (cf. (1.1)). We, of course, denote $h(D)$ as the operator e^{D^2}. We claim that if $f(z)$ is a complex polynomial of degree n, then

$$e^{D^2}[f(z)] = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2/4} f(z+t)dt. \tag{6.1}$$

By Taylor's Theorem, we know that

$$f(z+t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!} t^k,$$

and consequently,

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2/4} f(z+t)dt = \left(\sum_{k=0}^{n} \frac{f^{(k)}(z)}{k!} \right) \left(\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2/4} t^k dt \right).$$

When k is an odd integer, the function $e^{\frac{-t^2}{4}} t^k$ is odd, and the integral

$$\int_{-\infty}^{\infty} e^{-t^2/4} t^k dt = 0.$$

It follows that

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2/4} f(z+t) dt = \sum_{j=0}^{\lfloor n/2 \rfloor} \left(\frac{f^{(2j)}(z)}{2j!} \right) \left(\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-t^2}{4}} t^{2j} dt \right). \tag{6.2}$$

Substituting u for $t^2/4$, the integral on the right-hand side of (6.2) becomes

$$\int_{-\infty}^{\infty} e^{-t^2/4} t^{2j} dt = \int_{-\infty}^{\infty} e^{-u} 4^j u^{j-\frac{1}{2}} du = 4^j \int_{-\infty}^{\infty} e^{-u} u^{j+\frac{1}{2}-1} du.$$

In particular, the integral on the right-hand side of (6.2) is equal to $4^j \Gamma(j + \frac{1}{2})$. Thus, (6.2) becomes

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2/4} f(z+t) dt = \sum_{j=0}^{\lfloor n/2 \rfloor} \left(\frac{f^{(2j)}(z)}{2j!} \right) \left(\frac{1}{2\sqrt{\pi}} \right) \left(4^j \Gamma \left(j + \frac{1}{2} \right) \right).$$

By Legendre's Duplication Formula (cf. [125, p. 23]),

$$4^j \Gamma(j + \frac{1}{2}) = \frac{2^{2j}\sqrt{\pi}\Gamma(2j)}{2^{2j-1}\Gamma(j)} = \frac{2\sqrt{\pi}\Gamma(2j)}{\Gamma(j)}.$$

Of course, $\frac{\Gamma(2j)}{\Gamma(j)} = \frac{(2j-1)!}{(j-1)!}$, and thus,

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-t^2}{4}} f(z+t) dt = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{f^{(2j)}(z)}{j!} = \sum_{j=0}^{\infty} \frac{f^{(2j)}(z)}{j!} = e^{D^2}[f(z)].$$

From the integral representation of the linear operator e^{D^2} in (6.1), we can deduce several properties of e^{D^2}.

Proposition 6.11. *Let $f(z) \in \mathbb{C}[z]$ and let $z_0 \in \mathbb{C}$ be a fixed point. If* $\operatorname{Im} f(z_0+t) > 0$ *for all $t \in \mathbb{R}$, then* $\operatorname{Im}(e^{D^2}[f(z_0)]) > 0$. *If* $\operatorname{Re} f(z_0+t) > 0$ *for all $t \in \mathbb{R}$, then* $\operatorname{Re}(e^{D^2}[f(z_0)]) > 0$.

Proof. Fix $z_0 \in \mathbb{C}$. If $\operatorname{Im} f(z_0+t) > 0$ for all $t \in \mathbb{R}$, then $\operatorname{Im}(e^{-t^2/4} f(z_0+t)) > 0$ for all $t \in \mathbb{R}$. Hence,

$$\operatorname{Im}(e^{D^2}[f(z_0)]) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \operatorname{Im}(e^{-t^2/4} f(z_0+t)) dt > 0.$$

Similarly, if $\operatorname{Re} f(z_0+t) > 0$ for all $t \in \mathbb{R}$, then $\operatorname{Re}(e^{-t^2/4} f(z_0+t)) > 0$ for all $t \in \mathbb{R}$, and

$$\operatorname{Re}(e^{D^2}[f(z_0)]) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \operatorname{Re}(e^{-t^2/4} f(z_0+t)) dt > 0.$$

□

Definition 6.12. A real polynomial $f(x)$ is said to be *positive* if $f(x) > 0$ for all $x \in \mathbb{R}$. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is said to *preserve positivity* if $T[f(x)]$ is positive whenever $f(x)$ is positive.

Corollary 6.13. *The linear operator* $e^{D^2}:\mathbb{R}[x]\to\mathbb{R}[x]$ *preserves positivity.*

Proof. Let $f(x)$ be positive and let $x_0\in\mathbb{R}$. Since $f(x)$ is positive, and $x_0+t\in\mathbb{R}$ for all $t\in\mathbb{R}$, $f(x_0+t)>0$ for all $t\in\mathbb{R}$. By Proposition 6.11,

$$e^{D^2}[f](x_0)=\mathrm{Re}(e^{D^2}[f](x_0))>0.$$

Thus, e^{D^2} preserves positivity. □

We conclude this section with the following open problem.

Problem 6.14. Characterize all linear operators $T:\mathbb{R}[x]\to\mathbb{R}[x]$ which preserve positivity.

7. Location of Zeros

The results of the previous sections focused on the *number* of non-real zeros of real polynomials under the action of linear operators. Our goal in this section is to complement the above work and initiate the study of the *location* of zeros of complex polynomials under the action of linear operators. The open problems in this area of investigation appear to be very difficult (see, for example, Problem 1.3 and Problem 1.4 in Section 1). Here we confine our attentions to linear operators $T:\mathbb{C}[z]\to\mathbb{C}[z]$ which enjoy the Gauss-Lucas property (see Definition 7.1). One of our goals is to generalize the Gauss-Lucas Theorem (see Theorem 7.11).

7.1. **The Gauss-Lucas Theorem.**

Definition 7.1 (cf. [46])**.** A linear operator $T:\mathbb{C}[z]\to\mathbb{C}[z]$ is said to possess the *Gauss-Lucas property* if, whenever K is a convex region of $\mathbb{C}$ containing the origin and all the zeros of a complex polynomial $f(z)$, then all the zeros of $T[f](z)$ also lie in K.

The following classical result shows that the differential operator, $D=d/dz$, possesses the Gauss-Lucas property.

Theorem 7.2 (Gauss-Lucas Theorem [38])**.** *If $p(z)$ is a complex polynomial, then all the zeros of $p'(z)$ are located in the closed convex hull of the zeros of $p(z)$.*

The proof of the Gauss-Lucas Theorem follows from the following lemma and the fact that a convex hull is the intersection of half-planes.

Lemma 7.3. *If $f(z)$ is a complex polynomial with all of its zeros in a half-plane, then $f'(z)$ has all of its zeros in the same half plane.*

Proof. By means of the transformation $z\mapsto az+b$, we can map any half-plane to any other half-plane. Thus, it suffices to prove the lemma for the upper-half plane, $H_u=\{z:\mathrm{Im}(z)>0\}$. Suppose that $f(z)=c\prod_{k=1}^{n}(z-a_k)$, where $\mathrm{Im}\,a_k>0$ for $k=1,2,\ldots,n$. Consider the logarithmic derivative of $f(z)$,

$$\frac{f'(z)}{f(z)}=\sum_{k=1}^{n}\frac{1}{z-a_k}.$$

Now, for any $z\in\{z:\mathrm{Im}\,z\le 0\}$, it must be that $\mathrm{Im}(z-a_k)=\mathrm{Im}\,z-\mathrm{Im}\,a_k<0$, since $\mathrm{Im}\,a_k>0$. Hence, $\mathrm{Im}(\frac{1}{z-a_k})>0$ for each $k=1,2,\ldots,n$. Therefore,

$$\sum_{k=1}^{n} \frac{1}{z - a_k} \neq 0$$

whenever $\operatorname{Im} z \leq 0$. We conclude that $f'(z)$ cannot have any zeros in $\{z : \operatorname{Im} z \leq 0\}$. We can discount the case when $p'(z)$ and $p(z)$ share a zero, for then they will lie in the same half-plane. □

Remark 7.4. Theorem 7.2 shows that the differential operator, $D = d/dz$, possesses a property stronger than the Gauss-Lucas property; namely, all the zeros of $p'(z)$ are contained in *any* convex region K containing all the zeros of $p(z)$, and K need not contain the origin. The linear operator $\theta = xD$, however, possesses the Gauss-Lucas property without possessing this stronger requirement.

Definition 7.5. If $z = \alpha + i\beta \quad (\beta \neq 0)$ is a zero of a real entire function $f(z)$, then the *Jensen circle* of f is the circle centered at α with radius $|\beta|$.

Theorem 7.6 (Jensen's Theorem)**.** *If $f(z)$ is a real polynomial, then the non-real zeros of $f'(z)$ lie on or in some Jensen circle of $f(z)$.*

Proof. Let $f(z)$ be a real polynomial of degree n with m real and $2N$ non-real zeros. Then

$$f(z) = c \left[\prod_{k=1}^{m} (z - z_k) \right] \left[\prod_{j=1}^{N} (z^2 + 2a_j z + a_j^2 + b_j^2) \right],$$

where $z_k, a_j, b_j \in \mathbb{R}$. Suppose that $\hat{z} = a + bi$ is a non-real zero of $f'(z)$ which is not a zero of $f(z)$. Then

$$\frac{f'(\hat{z})}{f(\hat{z})} = \sum_{k=1}^{m} \frac{1}{\hat{z} - z_k} + \sum_{j=1}^{N} \frac{1}{\hat{z} - a_j - ib_j} + \frac{1}{\hat{z} - a_j + ib_j} = 0$$

and

$$0 = \operatorname{Im}\left(\frac{f'(\hat{z})}{f(\hat{z})}\right) = b \sum_{k=1}^{m} \frac{1}{|\hat{z} - z_k|^2} + \sum_{j=1}^{N} \frac{b - b_j}{|\hat{z} - a_j - ib_j|^2} + \frac{b + b_j}{|\hat{z} - a_j + ib_j|^2}. \tag{7.1}$$

Simplifying the right-hand side of (7.1) yields

$$\sum_{k=1}^{m} \frac{1}{|\hat{z} - z_j|^2} + 2 \sum_{j=1}^{N} \frac{(a - a_j)^2 + b^2 - b_j^2}{|\hat{z} - a_j - ib_j|^2 |\hat{z} - a_j + ib_j|^2} = 0.$$

If $(a - a_j)^2 + b^2 - b_j^2 > 0$ for all j, then we get that $\frac{f'(\hat{z})}{f(\hat{z})} > 0$, a contradiction. Thus, there is an $n \in \mathbb{N}$ such that $(a - a_n)^2 + b^2 \leq b_n^2$; that is, $a + ib = \hat{z}$ is in or on a Jensen circle of $f(z)$. □

Remark 7.7. If $f(z)$ is a hyperbolic polynomial and $T : \mathbb{C}[z] \to \mathbb{C}[z]$ has the Gauss-Lucas property, then the zeros of $T[f(z)]$ must be located in the convex hull of the zeros of f, which all lie on the real line. We conclude that if T has the Gauss-Lucas property, then T is a hyperbolicity preserver (cf. Definition 1.5).

The following example shows that while $D = d/dx$ has the Gauss-Lucas property, the immediate analog of the Hermite-Poulain Theorem (Theorem 1.12) for operators with the Gauss-Lucas property fails. That is, it is not true that for all $\gamma \in \mathbb{C}$, the linear operator $\gamma + D$ has the Gauss-Lucas property.

Example 7.8. Let $\gamma \neq 0$ be a complex number. Let $c = (32/15\gamma)^4$. Then the quartic

$$f(z) = z^4 - c$$

has zeros at $32/15\gamma, -32/15\gamma, 32i/15\gamma$ and $-32i/15\gamma$. Let T be the linear operator $\gamma + D$. Applying T, we get that $T[f(z)] = \gamma(z^4 - c) + 4z^3$. We calculate

$$\begin{aligned} T[f](-2c^{1/4}) &= \gamma(16c - c) - 32(c^{3/4}) \\ &= \gamma 15(32/15\gamma)^4 - 32(32/15\gamma)^3 \\ &= (1/\gamma^3)(32^4/15^3 - 32^4/15^3) = 0. \end{aligned}$$

Thus, $-2c^{1/4} = -2(32/15\gamma)$ is a zero of $T[f]$, which is outside the convex hull of the zeros of f.

7.2. **Generalizations of the Gauss-Lucas Theorem.** We wish to extend the Gauss-Lucas property to transcendental entire functions. However, in general, Definition 7.1 will not make sense, as it may not be possible to contain all of the possibly infinitely many zeros of a transcendental entire function $f(z)$ in a convex region. But by limiting ourselves to the zeros of the entire functions in the $\mathcal{L} - \mathcal{P}^*$ class, we are able to extend the Gauss-Lucas property to the $\mathcal{L} - \mathcal{P}^*$ class.

Definition 7.9. A linear operator $T : \mathcal{L} - \mathcal{P}^* \to \mathcal{L} - \mathcal{P}^*$ is said to possess the *extended Gauss-Lucas property*, if whenever $f(z) \in \mathcal{L} - \mathcal{P}^*$ has all of its zeros in a strip $K = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq M\}$ of $\mathbb{C}$, then all the zeros of $T[f](z)$ also lie in K.

Theorem 7.10 (Craven-Csordas [38, Theorem 3.11])**.** *Let $\varphi(z) = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k, \gamma_k > 0$, be a function of type* I *in the Laguerre-Pólya class (recall Definition 1.1), and let $f(z) \in \mathcal{L} - \mathcal{P}^*$, where the zeros of $f(z)$ all lie in the strip $K = \{z : |\operatorname{Im} z| \leq M\}$. Then the zeros of the entire function $\sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k f^{(k)}(z)$ all lie in K.*

Theorem 7.11. *Let $T : \mathbb{C}[z] \to \mathbb{C}[z]$ be a linear operator with the Gauss-Lucas property (cf. Definition 7.1), and suppose that we can express it in the form*

$$T = \sum_{k=0}^{m} Q_k(z) D^k, \tag{7.2}$$

where $Q_k(z) \in \mathbb{C}[z]$. Then, T has the extended Gauss-Lucas property.

Proof. Let $f(z) \in \mathcal{L} - \mathcal{P}^*$ with $f(z) = p(z)\varphi(z)$, where $p(z) \in \mathbb{R}[z]$ and $\varphi(z) \in \mathcal{L} - \mathcal{P}$. Suppose that the zeros of $f(z)$ lie in the strip $K = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq M\}$. Recall that we can uniformly approximate $\varphi(z)$ on compact subsets of $\mathbb{C}$ by a sequence of hyperbolic polynomials $g_n(z)$. It follows that $p(z)g_n(z) \to p(z)\varphi(z) = f(z)$ uniformly on compact subsets of $\mathbb{C}$. Since T has the Gauss-Lucas property, the zeros of $T[p(z)g_n(z)]$ all lie in the convex hull of the zeros of $p(z)g_n(z)$ for all n. But that convex hull must be contained the strip K, as the non-real zeros of $p(z)g_n(z)$ are the same as the non-real zeros of $f(z)$. Thus, for all n, the zeros of $T[p(z)g_n(z)]$

all lie in K. By Theorem 3.5, we know that $T[p(z)g_n(z)] \to T[f(z)]$ uniformly on compact subsets of $\mathbb{C}$. We can then apply Hurwitz's Theorem to conclude that the zeros of $T[f(z)]$ all lie in K. $\square$

In [38], Craven and Csordas examined the Gauss-Lucas Theorem in the context of multiplier sequences, and proved the following result.

Theorem 7.12 (Craven-Csordas [38, Theorem 2.8])**.** *Let $T = \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_k \geq 0$, be a multiplier sequence. If $0 \leq \gamma_k \leq \gamma_{k+1}$ for all k, then the associated linear operator T has the Gauss-Lucas property.*

Example 7.13. The requirement that $T = \{\gamma_k\}_{k=0}^{\infty}$ is a multiplier sequence in Theorem 7.12 is necessary, as if $f(x)$ is hyperbolic and $T[f(x)]$ has non-real zeros, then the non-real zeros of $T[f(x)]$ will lie outside a sufficiently small convex region of $\mathbb{C}$ containing the zeros of f and the origin. The requirement that $0 \leq \gamma_k \leq \gamma_{k+1}$ is also necessary, as the following example shows.

To prove that the sequence $T = \{\frac{1}{k!}\}_{k=0}^{\infty}$ is a multiplier sequence, we recall that if a polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ is hyperbolic, then its reverse (cf. Definition 1.40), $f^*(x) = \sum_{k=0}^{n} a_k x^{n-k}$, is also hyperbolic. Then by the Hermite-Poulain Theorem (Theorem 1.12),

$$f^*(D)[\frac{x^n}{n!}] = \sum_{k=0}^{n} a_k x^k = f(x)$$

is hyperbolic. Now, let $f(x) = x^4 - 1$, which has zeros at $1, -1, i$ and $-i$. The polynomial $T[f(x)] = \frac{1}{4!}x^4 - 1$ has zeros at $24, -24, 24i$ and $-24i$, which are certainly outside the convex hull of the zeros of $f(x)$ and the origin. Thus, T does not have the Gauss-Lucas property.

8. Appendix

8.1. **Miscellaneous Results.** We will now provide two proofs of the assertion (1.1a) in Section 1 (see Proposition 8.1); the second appears to be new. We also establish a generating relation for the Appell polynomials (see Proposition 8.3). We conclude this section with a uniqueness theorem for complex polynomials (see Proposition 8.4).

Proposition 8.1 (Section 1)**.** *If $h(y) = e^{-y^2}$ and $f(z) = e^{-\alpha z^2}$, then*

$$h(D)[f(z)] = \left(\sum_{k=0}^{\infty} \binom{2k}{k} \alpha^k \right) \exp\left(-\frac{\alpha z^2}{1 - 4\alpha} \right). \tag{8.1}$$

To expedite our presentation, we first establish the following lemma.

Lemma 8.2. *Let μ be a non-negative integer. Then for $0 < \alpha < 1/4$,*

$$\sum_{k=0}^{\infty} \frac{(2\mu + 2k)!}{k!(\mu + k)!} (-\alpha)^k = \frac{(2\mu)!}{\mu!} \frac{1}{(1 - 4\alpha)^{\mu + 1/2}}. \tag{8.2}$$

Proof. For convenience, we will use the *Pocchammer notation*:

$$(\beta)_n = \prod_{k=1}^{n} (\beta + k - 1) \quad (n \geq 1, \beta \neq 0),$$

where $(\beta)_0 = 1$. Then we can readily establish [125, p. 23] that

$$(\beta)_n = \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \quad (\beta > 0). \tag{8.3}$$

With the aid of the formula in (8.3), we obtain

$$\frac{(2\mu+2k)!}{(\mu+k)!} = \frac{(2\mu)!}{\mu!}\frac{(2\mu+1)_{2k}}{(\mu+1)_k} \quad (k = 0, 1, 2, \ldots). \tag{8.4}$$

Thus, using (8.4),

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{(2\mu+2k)!}{k!(\mu+k)!}(-\alpha)^k &= \frac{(2\mu)!}{\mu!}\sum_{k=0}^{\infty}\frac{(2\mu+1)_{2k}}{k!(\mu+1)_k}(-\alpha)^k \\
&= \frac{(2\mu)!}{\mu!}\sum_{k=0}^{\infty}\frac{(-\alpha)^k 2^{2k}}{k!}\left(\mu+\frac{1}{2}\right)\left(\mu+\frac{3}{2}\right)\cdots\left(\mu+\frac{2k-1}{2}\right) \\
&= \frac{(2\mu)!}{\mu!}\frac{1}{(1-4\alpha)^{\mu+1/2}}.
\end{aligned}$$

□

Proof of Proposition 8.1. (Method 1) Since the power series $\sum_{j=0}^{\infty} \frac{(-\alpha)^j}{j!} z^{2j}$ converges uniformly and absolutely on compact subsets of $\mathbb{C}$, term-by-term differentiation of this series is justified and whence, using (8.2), we obtain

$$\begin{aligned}
h(D)[f_\alpha(z)] &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}\frac{(2j)!}{(2j-2k)!}z^{2j-2k} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{j=k}^{\infty}\frac{(-\alpha)^j}{j!}\frac{(2j)!}{(2j-2k)!}z^{2j-2k} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sum_{\mu=0}^{\infty}\frac{(-\alpha)^{\mu+k}}{(\mu+k)!}\frac{(2\mu+2k)!}{(2\mu)!}z^{2\mu} \\
&= \sum_{\mu=0}^{\infty}\frac{(-\alpha)^{\mu}}{(2\mu)!}z^{2\mu}\sum_{k=0}^{\infty}\frac{(-\alpha)^k(2\mu+2k)!}{k!(\mu+k)!} \\
&= \sum_{\mu=0}^{\infty}\frac{(-\alpha)^{\mu}}{(2\mu)!}z^{2\mu}\frac{(2\mu)!}{\mu!}\frac{1}{(1-4\alpha)^{\mu+1/2}} \\
&= \frac{1}{\sqrt{1-4\alpha}}\sum_{\mu=0}^{\infty}\frac{(-\alpha)^{\mu}}{\mu!}\frac{z^{2\mu}}{(1-4\alpha)^{\mu}} \\
&= \left(\sum_{k=0}^{\infty}\binom{2k}{k}\alpha^k\right)\exp\left(-\frac{\alpha z^2}{1-4\alpha}\right) \quad (0 < \alpha < \frac{1}{4}),
\end{aligned}$$

where the absolute convergence of the indicated series allowed us to interchange the order of summation. □

Proof of Proposition 8.1. (Method 2) We first recall two standard formulae; the first from the theory of differential equations and the second from the theory of Fourier transforms:

$$e^{-D^2}\cos(zt) = e^{-t^2}\cos(zt) \quad \text{and} \tag{8.5}$$

$$e^{-\alpha z^2} = \frac{1}{\sqrt{\pi\alpha}}\int_0^\infty e^{-t^2/(4\alpha)}\cos(zt)dt \quad (\alpha > 0). \tag{8.6}$$

Now, by Leibniz's Rule, we can differentiate under the integral in (8.6). Hence by (8.5) and (8.6), we obtain

$$\begin{aligned} e^{-D^2}e^{-\alpha z^2} &= \frac{1}{\sqrt{\pi\alpha}}\int_0^\infty e^{-t^2/(4\alpha)}e^{-D^2}[\cos(zt)]dt \\ &= \frac{1}{\sqrt{\pi\alpha}}\int_0^\infty e^{-t^2/(4\alpha)}e^{-t^2}\cos(zt)dt \\ &= \frac{1}{\sqrt{1-4\alpha}}\exp\left(-\frac{\alpha z^2}{1-4\alpha}\right). \end{aligned}$$

The integral

$$\int_0^\infty e^{-t^2/(4\alpha)}e^{-t^2}\cos(zt)dt \tag{8.7}$$

can be evaluated by noting that $\cos(xt) = \operatorname{Re} e^{ixt}$ $(x \in \mathbb{R})$, and then completing squares. Here we used Mathematica. □

Proposition 8.3 (Remark 1.28)**.** *Let $f(x) = \sum_{k=0}^\infty \gamma_k x^k/k!$ be a real entire function and recall the definition of the n^{th} Appell polynomial,*

$$P_n(t) = \frac{1}{n!}\sum_{k=0}^n \binom{n}{k}\gamma_k t^{n-k} \qquad (n = 0, 1, 2, \ldots).$$

Then,

$$e^{xt}f(x) = \sum_{n=0}^\infty P_n(t)x^n.$$

Proof. Consider the power series

$$e^{xt}f(x) = \left(\sum_{k=0}^\infty \frac{x^k t^k}{k!}\right)\left(\sum_{j=0}^\infty \frac{\gamma_j x^j}{j!}\right).$$

Since e^{xt} and $f(x)$ are entire functions, we may take the Cauchy product and obtain

$$\begin{aligned}
e^{xt}f(x) &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{(xt)^{k-j}}{(k-j)!}\frac{\gamma_j x^j}{(j)!}\right)\\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\frac{t^{k-j}\gamma_j}{j!(k-j)!}\right)x^k\\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}\frac{t^{k-j}\gamma_j}{k!}\right)x^k\\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{j=0}^{k}\binom{k}{j}t^{k-j}\gamma_j\right)x^k\\
&= \sum_{k=0}^{\infty}P_k(t)x^k.
\end{aligned}$$

□

Proposition 8.4 (Proof of Proposition 3.4)**.** *If $f(z) \in \mathbb{C}[z]$ is a polynomial of degree n, then $f(z)$ is completely determined by $n+1$ distinct points.*

Proof. To prove this assertion, we first note that as a consequence of the Fundamental Theorem of Algebra, if a polynomial of degree n has $n+1$ distinct zeros, then it is identically zero. Now, let $f(z) = \sum_{k=0}^{n}\gamma_k z^k$ and $g(z) = \sum_{i=0}^{n}\delta_i z^i$ be two polynomials of degree n. Let $a_1, a_2, \ldots, a_{n+1} \in \mathbb{C}$ be $n+1$ distinct points such that $f(a_i), g(a_i) = b_i$ for some $b_i \in \mathbb{C}$ $(i = 1, 2, \ldots, n+1)$. Then $(f-g)(a_i) = 0$ for each i, and since $f - g$ is a polynomial of degree at most n, it is identically zero. Thus, $f(z) = g(z)$, and hence, a polynomial of degree n is determined by $n+1$ distinct points. □

8.2. **Perturbation Arguments in Conjunction with Hurwitz's Theorem.** We first recall the statement of Hurwitz's Theorem.

Theorem 8.5 (Hurwitz)**.** *Suppose that a sequence of analytic functions $\{f_k(z)\}_{k=0}^{\infty}$ converges to an analytic function $f(z)$ uniformly on compact subsets of $\mathbb{C}$, where $f(z)$ is not identically zero. If $z_0 \in \mathbb{C}$ is a zero of $f(z)$ of multiplicity m, then, for every sufficiently small neighborhood K of z_0, there exists an integer $N = N(K)$ such that K contains exactly m zeros of $f_n(z)$ (counting multiplicities) whenever $n \geq N$.*

As a consequence of Hurwitz's Theorem, we have the following corollary.

Corollary 8.6. *Let $f(z)$ be an entire function. Let $\{f_k(z)\}_{k=0}^{\infty}$ be a sequence of entire functions such that $f_k(z) \to f(z)$ uniformly on compact subsets of $\mathbb{C}$. Then there exists an $N \in \mathbb{N}$, such that, whenever $n \geq N$,*

$$Z_c(f(z)) \leq Z_c(f_n(z)),$$

where $Z_c(f(z))$ denotes the number of non-real zeros of $f(z)$, counting multiplicities.

Proof. (An Argument by Contradiction). Let M be the least positive integer such that, for all $N \in \mathbb{N}$,

$$Z_c(f(z)) \geq M > Z_c(f_n(z))$$

for some $n \geq N$. Let $\{z_1, z_2, \ldots, z_u\}$ be a set of distinct non-real zeros of f such that $\sum_{k=1}^{u} m(z_k) \geq M$, where $m(z_i)$ denotes the multiplicity of z_i. Let $\delta > 0$ be sufficiently small, where

$$2\delta < \min\left\{\min\{|\mathrm{Im}\, z_i| : i = 1, 2, \ldots, u\}, \min\{|z_i - z_j| : i \neq j\}\right\}.$$

By Hurwitz's Theorem, for each z_i, there is a radius $0 < \delta_i < \delta$ sufficiently small and an $N_i \in \mathbb{N}$ sufficiently large such that $f_n(z)$ and $f(z)$ have the same number of zeros in the closed disk $\overline{B(z_i, \delta_i)}$ whenever $n \geq N_i$. If we let $N \geq \max_{1 \leq j \leq u} N_j$, then for all $n \geq N$, $Z_c(f_n(x)) \geq M$, and thus we have arrived at a contradiction.

□

Finally, we give here a sufficient condition for uniform convergence on compact subsets.

Proposition 8.7 ([116, p. 20]). *Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a complex polynomial. Let $\{f_j(z)\}_{j=0}^{\infty}$ be a sequence of polynomials such that*

$$f_j(z) = \sum_{k=0}^{n} a_{k,j} z^k,$$

where

$$a_{k,j} \to a_k \quad (k = 0, 1, 2, \ldots).$$

Then, $f_j \to f$ uniformly on compact subsets of $\mathbb{C}$.

Proof. Let $K \subset \mathbb{C}$ be a compact set. Let $\epsilon > 0$. Since f is an entire function, there is a maximum $M = \max\{1, \sup_{z \in K} |z|\}$. We can then choose an integer N_k for each k such that

$$|a_k - a_{k,j}| < \frac{\epsilon}{(n+1)M^n}$$

whenever $j > N_k$. Set $N = \max\{N_0, N_1, \ldots, N_n\}$. Then, for $j > N$ and $z \in K$,

$$\begin{aligned} |f(z) - f_j(z)| &= |\sum_{k=0}^{n} a_k z^k - \sum_{k=0}^{n} a_{k,j} z^k| \\ &= \sum_{k=0}^{n} (a_k - a_{k,j}) z^k| \\ &\leq \sum_{k=0}^{n} |a_k - a_{k,j}||z|^k \\ &< \sum_{k=0}^{n} \frac{\epsilon}{(n+1)M^n} M^n = \epsilon. \end{aligned}$$

□

This combined with Hurwitz's Theorem gives us the following corollary on the perturbation of the coefficients of polynomials.

Corollary 8.8. *Let $f(z) = \sum_{k=0}^{n} a_k z^k$ be a complex polynomial. Let $\{f_j(z)\}_{j=0}^{\infty}$ be a sequence of polynomials such that*

$$f_j(z) = \sum_{k=0}^{n} a_{k,j} z^k,$$

where

$$a_{k,j} \to a_k \quad (k = 0, 1, 2, \ldots).$$

Then there exists an $N \in \mathbb{N}$ such that, whenever $j \geq N$,

$$Z_c(f_j(z)) \leq Z_c(f(z)),$$

where $Z_c(f(z))$ denotes the number of non-real zeros of $f(z)$.

We note here that if $\{f_j(z)\}_{j=0}^{\infty}$ is a sequence of polynomials such that, for each j,

$$f_j(z) = c \prod_{i=1}^{n} (z - z_{i,j}) = \sum_{k=0}^{n} a_{k,j} z^k,$$

and $z_{i,j} \to z_i \in \mathbb{C}$ for each i, then $a_{k,j} \to a_k$ for each k, where $c \prod_{i=1}^{n} (z - z_i) = \sum_{k=0}^{n} a_k z^k$. Thus, Corollary 8.8 is also a corollary on the perturbation of the zeros of polynomials.

8.3. **Properties of Functions in the Laguerre-Pólya Class.** To prove Theorem 1.28, we begin by establishing some properties of multiplier sequences.

Proposition 8.9. *Let $T = \{\gamma_k\}_{k=0}^{\infty}$ be a multiplier sequence (cf. Definition 1.26).*

(i) For all $k \in \mathbb{N}$ such that $\gamma_k, \gamma_{k+2} \neq 0$, we have that $\gamma_k \gamma_{k+2} \geq 0$.

(ii) If there exists an integer $m \geq 0$ such that $\gamma_m \neq 0$, then for any $n > m$ such that $\gamma_n = 0$, $\gamma_k = 0$ for all $k \geq n$.

(iii) The sign of γ_k when it is non-zero is either always the same or alternates.

(iv) Let $k \geq 0$ be an integer. If $\gamma_k, \gamma_{k+1}, \gamma_{k+2} \neq 0$, then $|\gamma_{k+2}| \leq \gamma_{k+1}^2 / |\gamma_k|$.

Proof. (i) Suppose that for some integer $k \geq 0$, $\gamma_{k+2}\gamma_k < 0$. If $f(x) = x^k(x^2 - 1)$, then the polynomial $T[f(x)] = x^k(x^2 + 1)$, has two non-real zeros. Thus, a contradiction, and hence, (i) follows.

(ii) Let $n \geq 1$ be an integer such that $\gamma_{n-1} \neq 0$ and $\gamma_n = 0$. Suppose that $\gamma_{n+1} \neq 0$. Let $f(x) = x^{n-1}(x^2 + 2x + 1)$. By (i), we know that $\gamma_{n+1}\gamma_{n-1} > 0$, and thus, $T[f(x)] = x^{n-1}(\gamma_{n+1}x^2 + \gamma_{n-1})$ has two non-real zeros. Now, suppose that $m \geq 2$ is an integer such that $\gamma_{n+m} \neq 0$ and, for all $k < m$, $\gamma_{n+k} = 0$. That is, $\gamma_{n-1} \neq 0, \gamma_{n+m} \neq 0$, and $\gamma_n, \gamma_{n+1}, \ldots, \gamma_{n+m-1} = 0$. Then let $f(x) = x^{n-1}(x + 1)^{m+1}$. This polynomial has no real zeros, but $T[f(x)] = x^{n-1}(\gamma_{n+m}x^{m+1} + \gamma_{n-1})$ can have at most two real zeros. However, $T[f(x)]$ is of degree $m + 1 \geq 3$, and thus,

$T[f(x)]$ has non-real zeros. Hence, (ii) follows.

(iii) This assertion follows from (i) and (ii).

(iv) We prove (iv) by contradiction. Suppose that $|\gamma_{k+2}| > \gamma_{k+1}^2/|\gamma_k|$. Then $\gamma_{k+2}\gamma_k > \gamma_{k+1}^2$. Let $f(x) = x^k(x^2 + 2x + 1)$. We obtain that

$$T[f(x)] = x^k(\gamma_{k+2}x^2 + 2\gamma_{k+1}x + \gamma_k). \tag{8.8}$$

The discriminant of the quadratic in (8.8) is $4\gamma_{k+1}^2 - 4(\gamma_{k+1}\gamma_k) < 0$. Thus, $T[f(x)]$ has non-real zeros, and this is the desired contradiction. □

Theorem 8.10 (Theorem 1.28, see also [103, p. 343])**.** *Let $T = \{\gamma_k\}_{k=0}^\infty$ be a multiplier sequence. If $f(x) = \sum_{k=0}^\infty a_k x^k/k! \in \mathcal{L} - \mathcal{P}$, then the function $T[f(x)] := \sum_{k=0}^\infty \gamma_k a_k x^k/k!$ represents an entire function, and also belongs to the Laguerre-Pólya class.*

Proof. We first establish that $T[f(x)]$ is an entire function. Let $N = \min\{k \in \mathbb{N} : \gamma_k \neq 0\}$. If there exists an $n \geq N$ such that $\gamma_n = 0$, then by (ii) of Proposition 8.9, $\gamma_k = 0$ for all $k \geq n$, and consequently, $T[f(x)]$ is a polynomial. Thus, we may assume that $\gamma_n \neq 0$ for all $n \geq N$. By (iv) of Proposition 8.9, for all $k \geq N$, $|\gamma_{k+2}| \leq \gamma_{k+1}^2/|\gamma_k|$. By induction, we obtain that

$$|\gamma_{k+2}| \leq \frac{|\gamma_{N+1}|^{k-N+2}}{|\gamma_N|^{k-N+1}}.$$

It follows that, for $k \geq N+2$,

$$|\gamma_k| \leq \frac{|\gamma_N|^{N+1}}{|\gamma_{N+1}|^N}\left(\frac{|\gamma_{N+1}|}{|\gamma_N|}\right)^k.$$

Thus,

$$\lim_{n\to\infty} |\gamma_k|^{1/n} \leq \frac{|\gamma_{N+1}|}{|\gamma_N|} < \infty. \tag{8.9}$$

Now, let $f(x) = \sum_{k=0}^\infty a_k x^k$ denote an entire function in the Laguerre-Pólya class. Since its radius of convergence is infinite,

$$\lim_{n\to\infty} |a_k|^{1/n} = 0.$$

By (8.9),

$$\lim_{n\to\infty} |a_k\gamma_k|^{1/n} = \lim_{n\to\infty} |a_k|^{1/n}|\gamma_k|^{1/n} = 0.$$

It follows that the power series $\sum_{k=0}^\infty a_k\gamma_k x^k$ is entire. This power series is precisely $T[f(x)]$, and hence $T[f(x)]$ is an entire function.

Now that we have established that $T[f(x)]$ is an entire function, let $g_n(x) = \sum_{k=0}^n \binom{n}{k} a_k x^k$ be the Jensen polynomials associated with f. By Theorem 2.3, each $g_n(x)$ is hyperbolic, and by T being a multiplier sequence, $T[g_n(x)]$ is also hyperbolic for each n. The Jensen polynomials, $h_n(x)$, associated with $T[f(x)]$, are

$$h_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} \gamma_k a_k x^k \quad (n = 1, 2, \ldots).$$

For each n, $h_n(x) = T[g_n(x)]$, and thus is hyperbolic. Since $T[f(x)]$ is entire, we can apply Theorem 2.2 to obtain that $h_n(\frac{x}{n}) \to T[f(x)]$ uniformly on compact subsets of $\mathbb{C}$ as $n \to \infty$. As the uniform limit of hyperbolic polynomials on compact subsets of $\mathbb{C}$, $T[f(x)] \in \mathcal{L} - \mathcal{P}$. $\square$

8.4. **A Theorem of Fisk.**

Theorem 8.11 (Theorem 3.11, see also [64, Theorems 6.1–6.3])**.** *Let $T : \pi(\Omega) \to \pi(\Omega)$ be a linear operator (cf. Notation 1.2).*

(i) Let $\Omega = [0, \infty)$. If for all polynomials $f \in \pi(\Omega)$, the zeros of f and $T[f]$ weakly interlace (cf. Definition 3.1) and $\deg f = \deg T[f] + 1$, then T is a (non-zero) scalar multiple of the derivative.

(ii) If we instead assume that $\deg f = \deg T[f]$, then there are constants a, b, c such that $T[f(x)] = af(x) + (bx + c)f'(x)$.

(iii) If for all polynomials $f \in \pi(\mathbb{R})$, the zeros of f and $T[f]$ weakly interlace and $\deg f = \deg T[f] - 1$, then there are constants a, b, c, where a and c have the same sign, such that $T[f(x)] = (b + ax)f(x) + cf'(x)$.

Proof. (i) Let m be an integer, $m \geq 2$. By assumption, the zeros of $T[x^m]$ must interlace with the zeros of x^m and $\deg T[x^m] = m - 1$. Therefore, $T[x^m]$ must have a zero of multiplicity $m-1$ at 0; that is, $T[x^m] = c_m x^{m-1}$ for some constant $c_m \neq 0$. We also know that $T[x^1] = c_1$ for some constant c_1, as $T[x]$ must be of degree zero. Since we have determined $T[x^m]$ for all $m \geq 1$, we only need to determine $T[1]$ to completely characterize the action of T on the vector space of polynomials. We note that since $T[x - 1] = c_1 - T[1]$ must be of degree zero, $T[1]$ must be a constant, say c_0. Acting on the quadratic $x^2 - 2\alpha x + \alpha^2 = (x - \alpha)^2$ ($\alpha > 0$), which has a double zero at α, the polynomial

$$T[x^2 - 2\alpha x + \alpha^2] = c_2 x - 2\alpha c_1 + \alpha^2 c_0$$

must have a zero at α. Thus, we have the relation $c_2 = 2c_1 - \alpha c_0$. But α was an arbitrary positive number, and we have that $c_0 = 0$. Now, for $m \geq 1$, let us apply T to the polynomial $x^m(x - 1)^2$. Since $x^m(x - 1)^2$ has a double zero at 1, it must be that

$$T[x^m(x - 1)^2] = x^{m-1}(c_{m+2}x^2 - 2c_{m+1}x + c_m)$$

has a zero at 1. Therefore, $c_{m+2} - 2c_{m+1} + c_m = 0$ for $m \geq 1$. Solving the recurrence relation back to $c_2 = 2c_1$, we get that $c_m = mc_1$ for each $m \geq 1$. Thus, $T[x^m] = mc_1 x^{m-1}$ for $m \geq 1$, and $T[1] = 0$. Hence, $T = c_1 D$ for some non-zero constant c_1.

(ii) For $m \geq 1$, the zeros of x^m and $T[x^m]$ must interlace and $\deg T[x^m] = m$. The polynomial $T[x^m]$ must have a zero of multiplicity at least $m - 1$ at zero, and thus it must be of the form $T[x^m] = a_m x^m - b_m x^{m-1}$ for some constants a_m, b_m. For

$m = 0$, $T[x^0] = T[1] = a_0$ for some constant a_0. We have characterized the action of T on the vector space of polynomials. Now, let us apply T to the quadratic $(x-\alpha)^2$, for some $\alpha > 0$. We get that

$$T[(x-\alpha)^2] = a_2x^2 - b_2x - \alpha a_1 x + \alpha b_1 + \alpha^2 a_0.$$

Since $(x-\alpha)^2$ has a double zero at α, it must be that $T[(x-\alpha)^2]$ has a zero at α, and we calculate

$$T[(x-\alpha)^2](\alpha) = \alpha^2(a_2 - 2a_1 + a_0) - \alpha(b_2 - 2b_1) = 0.$$

Since this is true for all $\alpha > 0$, we obtain that $a_2 - 2a_1 + a_0 = 0$ and $b_2 - 2b_1 = 0$. Now, for $m \in \mathbb{N}, m \geq 1$ and $\alpha > 0$, we apply T to the polynomial $x^m(x-\alpha)^2$ to compute

$$T[x^m(x-\alpha)^2] = x^m(a_{m+2}x^2 - 2\alpha a_{m+1}x + \alpha^2) - x^{m-1}(b_{m+2}x^2 - 2\alpha b_{m+1}x + \alpha^2 b_m.) \tag{8.10}$$

Since $x^m(x-\alpha)^2$ has a double zero at α, it follows that $T[x^m(x-\alpha)^2]$ has a zero at α, and by substituting α for x in (8.10), we obtain that

$$\alpha^{m+2}(a_{m+2} - 2a_{m+1} + a_m) + \alpha^{m+1}(b_{m+2} - 2b_{m+1} + b_m) = 0.$$

This holds for all $\alpha > 0$, and consequently, for each integer $m \geq 1$, we have that $a_{m+2} - 2a_{m+1} + a_m = 0$ and $b_{m+2} - 2b_{m+1} + b_m = 0$. Solving the recurrence equation for the b_m down to $2b_2 = b_1$, we obtain that $b_m = mb_1$ for each $m \geq 1$. For the a_m, solving the recurrence equation down to $a_m - 2a_{m-1} + a_{m-2} = 0$, we get the relation $a_m = ma_1 - (m-1)a_0 = (m)(a_1 - a_0) + a_0$ for each $m \geq 1$. Therefore,

$$\begin{aligned} T[x^m] = a_m x^m - b_m x^{m-1} &= a_0 x^m + (m)(a_1 - a_0)x^m - mb_1 x^{m-1} \\ &= a_0 x^m + (a_1 - a_0)x(x^m)' - b_1(x^m)'. \end{aligned}$$

We check that $T[1] = a_0 = a_0 1 + (a_1 - a_0)x \cdot 0 - b_1 \cdot 0$, and conclude that for all $f(x) \in \mathbb{R}[x]$, $T[f(x)] = a_0 f(x) + ((a_1 - a_0)x - b_1)f'(x)$.

(iii) An argument similar to the proof of (ii) shows this.

□

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