ON SEMIFREE SYMPLECTIC CIRCLE ACTIONS

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To Mom & Dad

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Abstract

In 1988, Dusa McDuff constructed the only known example of a non-Hamiltonian symplectic circle action with fixed points. On a Kähler manifold, any symplectic circle action with fixed points is Hamiltonian, so McDuff's construction provides an example of a symplectic manifold which cannot support a Kähler structure. We prove some results giving cohomological constraints on generalizing McDuff's construction, focusing on semifree symplectic circle actions on 6-manifolds with fixed surfaces.

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Preface

Consider the equatorial action of the circle group S^1 on the 2-sphere S^2. The orbits of the action are the circles of latitude and the fixed points are the poles. S^2 supports an oriented area measure called a symplectic structure which is invariant under the action. Assuming for simplicity that S^1 wraps the equator around itself just once provides an example of a semifree symplectic circle action. The study of such actions, especially on closed manifolds, illustrates the interplay between symplectic topology and equivariant cohomology. These actions have quite a different flavor than, on one hand, actions of compact semisimple groups and, on the other hand, toric varieties. The difference arises from the fact that most interesting symplectic actions, such as the one described above, are Hamiltonian, meaning that there are globally defined functions arising from conserved quantities à la Noether's theorem. The existence of such functions, called Hamiltonians, allows one to bring in the powerful machinery of Morse theory, both ordinary and equivariant, to compute many of the topological invariants of the manifold. For circle actions, there arises the possibility that no such globally defined function exists.

In this dissertation, we highlight some facets of semifree symplectic circle actions which are amenable purely to the techniques of equivariant cohomology. We focus on the problem of finding extra conditions so that the existence of fixed points, a topological condition, implies the existence of the aforementioned globally defined Hamiltonian function. Just the existence of fixed points is not sufficient as McDuff has constucted an example of a non-Hamiltonian symplectic S^1-action with fixed points.

In Chapter 1, we review some basic material needed for the sequel. Chapter 2 reviews equivariant cohomolgy and equivariant characteristic classes. Section 2.6

provides a new construction of a useful normal equivariant characteristic class which greatly simplifies later calculations. Chapter 3 forms the heart of the dissertation, reviewing the necessary ingredients for the construction of non-Hamiltonian symplectic actions and providing in sections 3.5 and 3.6 some theorems concerning possible generalizations of the McDuff example. Section 3.7 provides a new proof of a Tolman-Weitsman result on semifree symplectic circle actions with isolated fixed points. Finally, in Chapter 4 we add some observations about other cohomological conditons vis-à-vis Hamiltonian versus non-Hamiltonian.

Chapter 1

Some Basic Material

1.1 Symplectic Vector Spaces

Let W be an $2n$-dimensional real vector space.

Definition 1.1.1. A *symplectic structure* on W is a skew-symmetric, nondegenerate bilinear form Ω. The pair (W, Ω) is called a *symplectic vector space*.

Every symplectic vector space has a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ satisfying

$$\Omega(e_i, e_j) = 0,$$

$$\Omega(f_i, f_j) = 0,$$

$$\Omega(e_i, f_j) = \delta_{ij},$$

however, such a basis is far from unique. Not all linear subspaces of W are symplectic since Ω must restrict to a nondegenerate form.

Definition 1.1.2. The *symplectic complement* of a linear subspace V of W is defined by

$$V^\Omega = \{w \in W | \; \Omega(w, v) = 0 \text{ for all } v \in V\}.$$

Proposition 1.1.3. *For subspaces U and V,*

(i) $\dim V + \dim V^{\Omega} = \dim W$.

(ii) $(V^{\Omega})^{\Omega} = V$.

(iii) $U \subseteq V \iff V^{\Omega} \subseteq U^{\Omega}$.

(iv) V *is symplectic* $\iff V \cap V^{\Omega} = 0 \iff W = V \oplus V^{\Omega}$.

Definition 1.1.4. A subspace V is *isotropic* if $V \subseteq V^{\Omega}$, *coisotropic* if $V^{\Omega} \subseteq V$, and *Lagrangian* if $V = V^{\Omega}$.

Every 1-dimensional subspace is isotropic and every codimension 1 subspace is coisotropic. An example of a Lagrangian subspace is that spanned by $\{e_1, \ldots, e_n\}$.

Definition 1.1.5. A *complex structure* J on W is a linear endomorphism satisfying $J^2 = -1$.

Remark 1.1.6. The 2-dimensional subspace spanned by $\{e_i, f_i\}$ is symplectic. Define J on this subspace by $e_i \mapsto f_i$ and $f_i \mapsto -e_i$. Since $J^2 = -1$, it defines a complex structure. In fact, every symplectic vector space (W, Ω) has a complex structure J which is said to be *compatible* with Ω, meaning that $g(u, v) = \Omega(u, Jv)$ is a positive definite form. The construction of the operator J can be made canonical if one first chooses a positive definite form G on W. One gets $J = (AA^*)^{-\frac{1}{2}}A$ where A is a skew-symmetric endomorphism of W satisfying $\Omega(u, v) = G(Au, v)$. Also, $g(u, v) = G(AA^*u, v)$. See Cannas da Silva [Ca01] for details.

1.2 Symplectic Manifolds

Definition 1.2.1. Let M^{2n} be a manifold. A smooth choice of symplectic structure on each tangent space defines a 2-form ω on M. If ω is closed, then the pair (M, ω) is said to be a *symplectic manifold*.

The nondegeneracy condition on each tangent space implies that ω^n is nonzero and thus defines a volume form on M, whence M is orientable. For a closed manifold, ω is cannot be exact since otherwise ω^n would be exact and thus would have zero integral over M by Stoke's theorem.

The canonical example of a symplectic manifold is $\mathbb{R}^{2n}$ with symplectic form $\omega = \sum dx_i dy_i$. The complex version is $\mathbb{C}^n$ with symplectic form $\omega = \frac{i}{2} \sum dz_k d\bar{z}_k$ where $z_k = x_k + iy_k$. It turns out that locally, every symplectic manifold looks like the preceeding example.

Theorem 1.2.2 (Darboux). *Let M^{2n} be a symplectic manifold and $p \in M$. Then there is a chart $(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that $\omega = \sum dx_i dy_i$ on U.*

Definition 1.2.3. A *symplectomorphism* between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a diffeomorphism $\varphi : M_1 \to M_2$ such that $\varphi^*(\omega_2) = \omega_1$.

Darboux's theorem implies that there are no local symplectic invariants other than dimension. It is a special case of the much more general Weinstein theorem.

Theorem 1.2.4 (Weinstein [We71]). *Let ω_0 and ω_1 be symplectic forms on a manifold M agreeing on a closed submanifold N. Then there are neighborhoods U_0 and U_1 of N and a symplectomorphism $\phi : U_0 \to U_1$ satisfying $\phi^*(\omega_1) = \omega_0$ and $\phi|_N = 1_N$.*

The proof uses what is known as the *Moser trick* which was originally used to prove

Theorem 1.2.5 (Moser [Mo65]). *Let ω_0 and ω_1 be symplectic forms on a compact manifold M. Suppose that for $\omega_t = (1-t)\omega_0 + t\omega_1$, $0 \leq t \leq 1$, $[\omega_t] \in H^2(M)$ is independent of t, or equivalently, $[\omega_1] = [\omega_0]$. Then there is an isotopy $\rho_t : M \to M$ with $\rho_t^*(\omega_t) = \omega_0$ for all $0 \leq t \leq 1$.*

Definition 1.2.6. Symplectic forms ω_0 and ω_1 on M are *deformation equivalent* if there is a smooth family of symplectic forms ω_t, $0 \leq t \leq 1$. They are *isotopic* if $[\omega_t]$ is independent of t, and they are *strongly isotopic* if there is an isotopy $\rho_t : M \to M$ with $\rho_1^*(\omega_1) = \omega_0$.

Thus,

$$\text{Strongly Isotopic} \Rightarrow \text{Isotopic} \Rightarrow \text{Deformation Equivalent}.$$

Moser's theorem can be used to prove that

$$\text{Isotopic} \Rightarrow \text{Strongly Isotopic}.$$

Remark 1.2.7. Since locally, any closed form is exact, locally, $\omega = -d\theta$, and θ is called a *symplectic potential*. In Darboux coordinates, $\theta = \sum y_i dx_i$. The sign convention comes from the construction of symplectic forms on cotangent bundles which is motivated by physics. For 1-dimensional mechanics, the phase space is $T^*\mathbb{R} \cong \mathbb{R}^2$ with coordinates $(q, p) = (position, momentum)$ and symplectic form $dqdp$. In this case, there is a global symplectic potential called the *tautological 1-form* $\theta = pdq$.

1.3 S^1-Actions

Given an S^1-action on M, there is an infinitesimal action of the Lie algebra $\mathfrak{h}$ of S^1 on $C^\infty(M)$ defined for $X \in \mathfrak{h}$ by

$$X(f)(x) = \frac{d}{dt}\Big|_{t=0} f(\exp(-tX) \cdot x).$$

Similarly, one can define an operator L_X on the sections s of any tensor bundle over M by

$$L_X(s) = \frac{d}{dt}\Big|_{t=0} \exp(-tX) \cdot s.$$

On forms, L_X is a derivation of degree 0. Along with the exterior derivative d and interior multiplication i_X, one has the *Cartan homotopy formula*

$$L_X = di_X + i_X d. \tag{1.3.1}$$

Definition 1.3.1. For a symplectic action, $L_X\omega = 0$, so from the above formula, $i_X\omega$ is a closed 1-form. If $i_X\omega$ is exact, then the action is called *Hamiltonian*.

Definition 1.3.2. An action is *almost free* if every isotropy subgroup is finite. Every S^1-action is almost free on the complement of its fixed point set. An action is *semifree* if every isotropy group is either trivial or the whole group.

Example 1.3.3. Take S^2 with symplectic form $d\phi d\cos\theta$. The S^1-action is given by rotation about the polar axis where north pole is defined by $\cos\theta = 1$ and the south pole is defined by $\cos\theta = -1$. We are abusing notation here as S^2 cannot be parametrized by one chart.

Example 1.3.4. Take $T^2 = S^1 \times S^1$ with symplectic form $d\phi d\theta$. The S^1-action is given by rotation on the first factor.

The first example is a Hamiltonian action and has fixed points. It would be nice to specify to what extent Hamiltonian actions are characterized by the existence of fixed points. The following are some results in this direction:

- A symplectic S^1-action on a Kähler manifold is Hamiltonian iff it has a fixed point. [Fr59]
- A symplectic S^1-action on a manifold whose cohomology algebra satisfys the weak Lefschetz condition, namely, $\wedge\omega^{n-1} : H^1 \xrightarrow{\simeq} H^{2n-1}$, is Hamiltonian iff it has a fixed point. [On88]
- A symplectic S^1-action on a symplectic 4-manifold is Hamiltonian iff it has a fixed point. [Mc88]
- A semifree symplectic S^1-action with isolated fixed points is Hamiltonian. [TW00]

Since every closed form is locally exact, every symplectic action is locally Hamiltonian. However, McDuff [Mc88] has given an example of a semifree symplectic S^1-action on a 6-manifold with fixed point set consisting of 2-tori which is not globally Hamiltonian.

1.4 Hamiltonians and Poisson Brackets

Definition 1.4.1. Given a vector field X on M, if $i_X\omega$ is exact, then $i_X\omega = dh$ for some smooth function h on M. Such a function is called a *Hamiltonian* of X. It is uniquely defined up to the addition of a locally constant function. If X is the infinitesimal generator of a Hamiltonian S^1-action, then h is called a *moment map*. By the nondegeneracy of ω, given any smooth function f, there is a vector field X_f such that $i_{X_f}\omega = df$. X_f is called the *symplectic gradient* of f.

In the above example of the S^1-action on S^2, the infinitesimal action is given by the vector field X_ϕ. Since $i_{X_\phi}\omega = d\cos\theta$, the Hamiltonian is $z = \cos\theta$. For the S^1-action on T^2, $i_{X_\phi}\omega = d\theta$ and even though locally there is a well-defined angle function θ on T^2, globally, one does not exist.

Definition 1.4.2. An *almost complex structure* J on a manifold M^{2n} is a smooth choice of complex structure on each tangent space. This is equivalent to a reduction of the structure group of TM from $O(2n)$ to $U(n)$.

Remark 1.4.3. Given a Riemannian metric on a symplectic manifold (M, ω), one can construct an almost complex structure J compatible with ω, meaning that $g(u, v) = \omega(u, Jv)$ is a Riemannian metric. This construction is canonical since it is canonical pointwise by Remark 1.1.6 and since there is a smooth square root on the bundle $\mathrm{Hom}(TM, TM)$. (ω, g, J) is called a *compatible triple.* One can use the metric g to define the *metric gradient* operator *grad*. There is a relationship between the metric gradient and the symplectic gradient

$$JX_f = \mathrm{grad} f$$

which, in particular, implies that $g(X_f, \mathrm{grad} f) = 0$.

Consider $\mathbb{R}^2$ with $\omega = dqdp$. Let $H = \frac{p^2}{2m} + V(q)$. This is the standard setup for motion in one dimension with potential V. Hamilton's equations give the time evolution of q, position, and p, momentum,

$$\frac{dq}{dt} = \frac{\partial H}{\partial p},$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

The symplectic and metric gradients are, respectively,

$$X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}$$

and

$$\text{grad}H = \frac{\partial H}{\partial q}\frac{\partial}{\partial q} + \frac{\partial H}{\partial p}\frac{\partial}{\partial p}.$$

One can verify that $JX_H = \text{grad}H$ where J is defined by $J(\frac{\partial}{\partial q}) = \frac{\partial}{\partial p}$ and $J(\frac{\partial}{\partial p}) = -\frac{\partial}{\partial q}$.

For S^1, the canonical example of a Hamiltonian action arises from multiplication in $\mathbb{C}$. Recall that the symplectic structure on $\mathbb{C}$ is given by $\frac{i}{2}dzd\bar{z}$. For the action defined by $z \mapsto e^{i\lambda\theta} \cdot z$, the infinitesimal action and moment map are

$$X_h = i\lambda\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right) \tag{1.4.1}$$

and

$$h = -\frac{1}{2}\lambda|z|^2 \tag{1.4.2}$$

where λ is an integer called the *weight* of the action. Through the equations

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

one gets

$$X_h = \lambda\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)$$

which is the infinitesimal action of a speed λ rotation about the origin in $\mathbb{R}^2$. By speed λ, it is meant that the circle wraps around the origin λ times, as is clear from the action given above.

The algebra of smooth functions $C^\infty(M)$ on a symplectic manifold M has a Lie algebra structure given by the *Poisson bracket* which is defined by

$$\begin{aligned}\{f,g\} &= \omega(X_f, X_g)\\ &= i_{X_f}\omega(X_g)\\ &= df(X_g)\\ &= X_g f.\end{aligned}$$

On $(\mathbb{R}^2, dqdp)$, the Poisson bracket has the form

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}.$$

Hamilton's equations can be recast as

$$\begin{aligned}\frac{dq}{dt} &= \{q, H\},\\ \frac{dp}{dt} &= \{p, H\}.\end{aligned}$$

Remark 1.4.4. This formulation of mechanics is crucial to understanding the relationship between the classical and quantum theories. For example, Heisenberg's uncertainty principle is derived from the fact that $[Q, P] \neq 0$ for the quantum position, Q, and momentum, P, observables. Classicaly, this is detected by $\{q, p\} \neq 0$.

1.5 The Equivariant Darboux Theorem

In [Fr59], Frankel shows that the fixed point components of a symplectic S^1-action are sympletic. Given a symplectic S^1-manifold M, for $p \in F$ a component of the fixed point set, it is clear that T_pF and N_pF, the *normal space* of F at p, are symplectic subspaces of T_pM and $T_pM = T_pF \oplus N_pF$. Also, the linear action of S^1 on N_pF is symplectic. There is an extension of this to charts by an equivariant version of the Darboux theorem for the neighborhood of a fixed point due to Weinstein [We77].

Theorem 1.5.1 (Equivariant Darboux)**.** *Let (M^{2n}, ω) be a symplectic S^1-manifold with $p \in F^{2k}$ a component of the fixed point set. Then there is an S^1-equivariant symplectic chart $\phi : (\mathbb{R}^{2n}, \sum dx_i dy_i) \to (M, \omega)$ centered at p where $\mathbb{R}^{2n}$ has a symplectic representation of S^1. The subspace $V^{2k} \subseteq \mathbb{R}^{2n}$ where the representation is trivial provides Darboux coordinates on F under ϕ.*

Since the symplectic representation of S^1 on N_pF must preserve a compatible almost complex structure on N_pF, the representation must also be complex and thus must split into a sum of 1-dimensional complex representations, $N_pF = \bigoplus_{i=1}^k E_i$. Each representation E_i has the form given by (1.4.1). The associated local moment map (1.4.2) yields a local interpretation of the fixed point set as the locus of attraction or repulsion of an harmonic oscillator potential depending on whether λ is negative or positive. Such a splitting does not exist at the bundle level, but we do have a splitting $NF = \bigoplus_\lambda E_\lambda$ over the eigenvalues λ of the representation. In the case of a semifree action, we have $NF = E_+ \oplus E_-$ where E_+ is the (+1)-eigenbundle and E_- is the (-1)-eigenbundle.

1.6 Morse Theory

Definition 1.6.1. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if the critical points p of f are *nondegenerate*, meaning that the *Hessian matrix* $H_p(f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is nonsingular.

Nondegeneracy does not depend on the choice of coordinates and, furthermore, implies that the critical points are isolated.

Definition 1.6.2. The *index* $\lambda_p(f)$ of a critical point p is the number of negative eigenvalues of $H_p(f)$. It also does not depend on the choice of coordinates.

The Betti numbers of M, $b_i(M)$, are constrained by the numbers of critical points of each index, $c_i(f)$, by the *Morse inequalilites* (1.6.1). Let

$$M_t(f) = \sum_p t^{\lambda_p(f)}$$
$$= \sum_i c_i(f) \cdot t^i$$

be the *Morse counting series* and

$$P_t(M) = \sum b_i(M) \cdot t^i$$

be the *Poincaré series.* Then there is a polynomial $R(t)$ with nonnegative coefficients satisfying

$$M_t(f) - P_t(M) = (1 + t) \cdot R(t). \tag{1.6.1}$$

Definition 1.6.3. A Morse function is *perfect* if $R(t) = 0$, whence $b_i(M) = c_i(f)$.

A CW-approximation to M can be built inductively from a Morse function by attaching a k-disk for each critical point of index k.

Bott [Bo54] extended Morse theory to include functions, called *Morse-Bott* functions, whose critical point sets consist of submanifolds.

Definition 1.6.4. A *nondegenerate critical submanifold* is a submanifold C on which $df = 0$ and $H_p(f)$ is nonsingular when restricted to the normal space N_pC.

The index $\lambda_C(f)$ of a nondegenerate critical submanifold C is the number of negative eigenvalues of this restriction. From the splitting $NC = N_+C \oplus N_-C$ of the normal bundle into positive and negative eigenbundles of $H_C(f)$, the index is well-defined. This is a generalization of the isolated critical point case. The *Morse-Bott inequalities* are now given by

$$M_{t,C}(f) - P_t(M) = (1+t) \cdot R(t)$$

where $R(t)$ is again a polynomial with nonnegative coefficients and

$$M_{t,C}(f) = \sum_C P_t(C) \cdot t^{\lambda_C(f)}$$

is the *Morse-Bott counting series.*

We again have an inductive construction of a CW-approximation from the critical point set, but we now attach the disk bundle of the negative normal bundle $N_{-}C$ for each critical submanifold C.

The upshot of this digression into Morse-Bott theory is that a Hamiltonian S^1-manifold supports a Morse-Bott function, the Hamiltonian, whose critical point set is exactly the fixed point set of the action. Moreover, as the fixed point set consists of symplectic submanifolds, the Morse-Bott indicies are all even, whence the Hamiltonian is perfect.

Chapter 2

Equivariant Characteristic Classes

2.1 The Borel Construction

Let G be a compact, connected Lie group. Given a principal G-bundle $P \to B$ and a G-space F, one can form the associated bundle

$$\begin{array}{ccc} F & \longrightarrow & P \times_G F \\ & & \downarrow \\ & & B \end{array}$$

where the free action of G on $P \times F$ is given by $g \cdot (p, f) = (p \cdot g^{-1}, g \cdot f)$. Since a G-bundle is completely determined by its transition functions, any G-bundle can be realized as an associated bundle of some principal G-bundle. By a construction of Milnor [Mi56], there is a universal principal G-bundle $EG \to BG$ where EG is contractible. Any principal G-bundle $P \to B$ can be realized as the pullback bundle of some map $f : B \to BG$. Maps homotopic to f induce isomorphic bundles. BG is called the *classifying space* of G.

Definition 2.1.1. Given a G-manifold M, the associated bundle to the universal G-bundle

$$\begin{array}{ccc} M & \longrightarrow & EG \times_G M \\ & & \downarrow \\ & & BG \end{array}$$

is called the *Borel construction*.

Definition 2.1.2. The ordinary cohomology of $M_G = EG \times_G M$ is called the *equivariant cohomology* of M,

$$H^*_G(M) = H^*(M_G).$$

Remark 2.1.3. We take all cohomology with coefficients in $\mathbb{R}$.

If the action is free, then $H^*_G(M) = H^*(M/G)$. On the other hand,

$$H^*_G(pt) = H^*(BG).$$

The Serre spectral sequence for the fibration $M_G \to BG$ is a guide to compute $H^*_G(M)$, though in practice such a computation may be extremely difficult, if not intractable. Recall that $E_2^{p,q} = H^p(BG) \otimes H^q(M)$. The equivariant cohomology functor H^*_G satisfies all but the dimension axiom of the Eilenberg-Steenrod axioms, so there is a Mayer-Vietoris sequence for invariant sets which facilitates an alternate method of calculation of $H^*_G(M)$.

It is useful for certain constructions to work with a finite dimensional smooth *N-approximation*

$$\begin{array}{ccc} M & \longrightarrow & M_{G,N} = EG_N \times_G M \\ & & \downarrow \\ & & BG_N \end{array}$$

to the Borel construction where $EG_N \to BG_N$ is an *N-universal G-bundle*, namely, a principal G-bundle with $(N-1)$-connected total space. $H^*_{G,N}(M) = H^*(M_{G,N})$ is an approximation to $H^*_G(M)$ and we have

$$\begin{aligned} EG &= \varinjlim EG_N, \\ BG &= \varinjlim BG_N, \\ M_G &= \varinjlim M_{G,N}, \\ H^*_G(M) &= \varprojlim H^*_{G,N}(M). \end{aligned} \tag{2.1.1}$$

2.2 The Weil Model

There is an equivariant form of the de Rham theorem. The *Weil model* gives an appropriate *differential graded algebra, DGA,* analogous to the de Rham algebra of forms. Denote the Lie algebra of S^1 by $\mathfrak{h}$. The *Weil algebra*

$$\begin{aligned} W\mathfrak{h} &= \Lambda\mathfrak{h}^* \otimes S\mathfrak{h}^* \\ &= \mathbb{R}[\theta, u]/(\theta^2) \end{aligned}$$

has generator θ of $\Lambda\mathfrak{h}^*$ of degree 1 and generator u of $S\mathfrak{h}^*$ of degree 2. The differential D on $W\mathfrak{h}$ is defined by $D\theta = u$ and $Du = 0$. As $\{1, u, u^2, \dots, \theta u, \theta u^2, \dots\}$ is a basis for $W\mathfrak{h}$ and $u^k = D(\theta u^{k-1})$, $W\mathfrak{h}$ is acyclic. One should think of $W\mathfrak{h}$ as an algebraic model of $ES^1 = S^\infty$; θ corresponds to the connection 1-form for the bundle $ES^1 \to BS^1$ and u corresponds to the curvature 2-form which represents the pullback of the universal first Chern class in $H^2(BS^1)$.

There is an action of $\mathfrak{h}$ on $W\mathfrak{h}$ given by

$$i_X\theta = 1,$$

$$i_X u = 0,$$

$$L_X = i_X D + D i_X,$$

where X is a generator of $\mathfrak{h}$. Elements in a DGA that satisfy $i_X a = L_X a = 0$ are called *basic*. Consider a principal S^1-bundle $P \to B$. Any form $\eta \in \Omega^* P$ which is *invariant*, $L_X\eta = 0$, and *horizontal*, $i_X\eta = 0$, must be a pullback of some form $\tilde{\eta} \in \Omega^* B$, hence the term. Thus, $(W\mathfrak{h})_{bas} = S\mathfrak{h}^*$.

The Weil model $\Omega^*_{\mathfrak{h}}(M)$ is the DGA of basic elements of $\Omega^* M \otimes W\mathfrak{h}$. The differential is given by $\mathcal{D} = d \otimes 1 + 1 \otimes D$. The action of $\mathfrak{h}$ is defined by $\mathcal{I}_X = i_X \otimes 1 + 1 \otimes i_X$ and $\mathcal{L}_X = L_X \otimes 1 + 1 \otimes L_X$. Following the treatment of Atiyah-Bott [AB84], any $\eta \in \Omega^* M \otimes W\mathfrak{h}$ can be written as

$$\eta = \sum_j \alpha_j \otimes u^j + \sum_k \beta_k \otimes \theta u^k$$

where $j, k \in \mathbb{N}$ and $\alpha_j, \beta_k \in \Omega^* M$. So η is basic if

$$\mathcal{I}_X \eta = \sum_j i_X \alpha_j \otimes u^j + \sum_k \left(i_X \beta_k \otimes \theta u^k + (-1)^{\deg \beta_k} \beta_k \otimes u^k \right) = 0$$

and

$$\mathcal{L}_X \eta = \sum_j L_X \alpha_j \otimes u^j + \sum_k L_X \beta_k \otimes \theta u^k = 0$$

which gives

$$i_X \alpha_j + (-1)^{\deg \beta_j} \beta_j = 0$$

and

$$L_X \alpha_j = 0.$$

Remark 2.2.1. The Weil model should be thought of as an equivariant de Rham model of the Borel construction. See Guillemin-Sternberg [GS99].

2.3 The Cartan Model

Consider the endomorphism $\gamma = i_X \otimes \theta$ of $\Omega^*_{\mathfrak{h}}(M)$. Since $\gamma^2 = 0$ we have automorphisms $e^{\gamma} = 1 + \gamma$ and $e^{-\gamma} = 1 - \gamma$. Since for $\eta \in \Omega^*_{\mathfrak{h}}(M)$

$$\gamma(\eta) = \sum_j (-1)^{\deg \alpha_j} i_X \alpha_j \otimes \theta u^j,$$

we get

$$e^{-\gamma}(\eta) = \sum_j \alpha_j \otimes u^j,$$

and so

$$\begin{aligned} \operatorname{im}(e^{-\gamma}) &= \Omega^*_{inv} M \otimes S\mathfrak{h}^* \\ &\simeq \Omega^*_{inv} M[u]. \end{aligned}$$

Thus, $e^{-\gamma}$ yields an isomorphism $\Omega^*_{\mathfrak{h}}(M) \simeq \Omega^*_{inv} M[u]$ called the *Mathai-Quillen isomorphism* [MQ86]. $\Omega^*_{inv} M[u]$ is called the *Cartan model* and has differential defined by $d_X = e^{-\gamma} \mathcal{D} e^{\gamma}$. On a basis $\{\alpha \otimes 1, 1 \otimes u\}$ we get

$$\begin{aligned} d_X \alpha &= d\alpha - i_X \alpha \cdot u, \\ d_X u &= 0. \end{aligned} \tag{2.3.1}$$

By (2.3.1), an element $\zeta = \sum_j \alpha_j \otimes u^j \in \Omega^*_{inv}M[u]$ is *equivariantly closed*, $d_X(\zeta) = 0$, iff

$$d\alpha_{j-1} = i_X \alpha_j \text{ for } i \geq 1. \tag{2.3.2}$$

Remark 2.3.1. If (M, ω) is a symplectic S^1-manifold, then the action is Hamiltonian iff ω has an equivariantly closed extention $\omega^\# = \omega + \mu \cdot u$ in the Cartan model where $\mu : M \to \mathfrak{h}^*$ is the equivariant moment map. In the Hamiltonian case, we would also obtain an extension of ω in the Weil model

$$\begin{aligned}\tilde{\omega} &= e^\gamma(\omega^\#) \\ &= \omega \otimes 1 + \mu \otimes u + d\mu \otimes \theta \\ &= \omega \otimes 1 + \mathcal{D}(\mu \otimes \theta).\end{aligned} \tag{2.3.3}$$

2.4 Equivariant Localization

Let M be a closed G-manifold. The equivariant cohomology $H^*_G(M)$ is an $H^*(BG)$-module under the pullback π^*_G of $M \xrightarrow{\pi} pt$. The *equivariant Gysin homomorphism*, or *equivariant pushforward*, of π

$$\pi^G_! : H^*_G(M) \to H^*(BG)$$

can be interpreted as *integration over the fibre*

$$\pi^G_!(\eta) = \int_M \eta$$

in the Borel construction. Let F be the fixed point set of the action. The equivariant pushforward $i_!^G$ of $F \xrightarrow{i} M$ satisfies

$$i_G^* i_!^G(1) = e(\nu_F)$$

where $e(\nu_F)$ is the equivariant Euler class of the normal bundle ν_F of F in M. If $e(\nu_F)$ is invertible in some localization $\mathcal{S}^{-1}H_G^*(F)$, then

$$\frac{i_G^*}{e(\nu_F)}$$

will be an inverse to $\mathcal{S}^{-1}H_G^*(F) \xrightarrow{i_!^G} \mathcal{S}^{-1}H_G^*(M)$. This will indeed be so if $\mathcal{S}$ is generated by the restrictions of $e(\nu_F)$ to a point p_i in each component F_i of F. This is the content of the *localization theorem*. The diagram

$$\begin{array}{ccc} \mathcal{S}^{-1}H_G^*(F) & \xleftarrow{\frac{i_G^*}{e(\nu_F)}} & \mathcal{S}^{-1}H_G^*(M) \\ & \searrow^{\pi_!^G} & \downarrow{\scriptstyle \pi_!^G} \\ & & \mathcal{S}^{-1}H_G^*(pt) \end{array}$$

provides the integration formula

$$\begin{aligned} \int_M \eta|_M &= \int_F \frac{\eta|_F}{e(\nu)} \\ &= \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)} \end{aligned} \tag{2.4.1}$$

where $\eta \in H_G^*(M)$ and the F_i are the components of F. The above treatment is by Atiyah-Bott [AB84].

Let M be a closed G-manifold. The equivariant cohomology $H_G^*(M)$ is an $H^*(BG)$-module. Let F be the fixed point set of the action. We have the following exact sequence of $H^*(BG)$-modules

$$H^*_G(M, M-F) \to H^*_G(M) \to H^*_G(M-F)$$

Let $i^*_{G,k}$ be the pullback of the inclusion of a point p_k in a component F_k of F. If we localize at the multiplicative set $\mathcal{S}$ generated by the pullbacks $i^*_{G,k}(\eta_k)$ of the equivariant Euler class η_k of F_k, we end up with an isomorphism of $\mathcal{S}^{-1}H^*(BG)$-modules

$$\mathcal{S}^{-1}H^*_G(M, M-F) \to \mathcal{S}^{-1}H^*_G(M)$$

since $H^*_G(M-F)$ is $\mathcal{S}$-torsional. We can choose the support of η close to F, so the restriction of η to a suitable open subset of $M-F$ is zero.

By excision, we obtain the exact sequence of $H^*(BG)$-modules

$$H^*_G(DF, SF) \to H^*_G(DF) \to H^*_G(SF)$$

where SF and DF are the normal sphere and disk bundles of F, respectively. Using excision again to shrink the support of η and then localizing at $\mathcal{S}$ yields an isomorphism of $\mathcal{S}^{-1}H^*(BG)$-modules

$$\mathcal{S}^{-1}H^*_G(DF, SF) \to \mathcal{S}^{-1}H^*_G(DF).$$

Since F is a deformation retract of DF, we have the localization isomorphism

$$\mathcal{S}^{-1}H^*_G(M) \to \mathcal{S}^{-1}H^*_G(F).$$

2.5 Equivariant Characteristic Classes

Let G and S be compact, connected Lie groups and M and P be S-manifolds.

Definition 2.5.1. A principal G-bundle $P \xrightarrow{\pi} M$ is *S-equivariant* if $s(gp) = g(sp)$ and $\pi(sp) = s\pi(p)$ for all $g \in G$, $s \in S$ and $p \in P$.

We get a principal G-bundle $P_S \xrightarrow{\pi_S} M_S$ of Borel constructions which is functorial. Let $ES \xrightarrow{\rho} BS$ be the universal principal S-bundle and let $ES_N \xrightarrow{\rho_N} BS_N$ be a finite dimensional smooth approximation. The diagrams of bundles

$$\begin{array}{ccc} P & \longrightarrow & P_S \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_S} \\ M & \longrightarrow & M_S \end{array}$$

and

$$\begin{array}{ccc} ES \times P & \longrightarrow & P_S \\ \downarrow & & \downarrow \\ ES \times M & \longrightarrow & M_S \end{array}$$

have approximations

$$\begin{array}{ccc} P & \longrightarrow & P_{S,N} = ES_N \times_S P \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_{S,N}} \\ M & \longrightarrow & M_{S,N} = ES_N \times_S M \end{array}$$

and

$$\begin{array}{ccc} ES_N \times P & \longrightarrow & P_{S,N} \\ \downarrow & & \downarrow \\ ES_N \times M & \longrightarrow & M_{S,N} \end{array}$$

can be used to show that the equivariant characteristic classes of the bundle π_S are inverse limit classes of ordinary characteristic classes of the bundles $\pi_{S,N}$.

Suppose that M^{2n} is almost complex, that $G = U(n)$, and that S preserves the almost complex structure. If the S-action on M is trivial, then S-equivariance implies that S commutes with G in each fibre of π. Furthermore, $M_S = BS \times M$ and so $H^*_S(M) = H^*(BS) \otimes H^*(M)$. Thus, in this case, the *total Chern class* $c(\pi_S)$ must satisfy

$$c(\pi_S) = c(\pi) \otimes c(\rho).$$

The total Chern class $c(M_{S,N}) \in H^*(M_{S,N}) = H^*_{S,N}(M)$ restricts to $c(M) \in H^*(M)$ under $M \to M_{S,N}$ and for the fixed point set F of the S-action on M restricts to

$$c(F_{S,N}) \cdot c(\nu_{F_{S,N}}) \in H^*(F_{S,N}) = H^*_{S,N}(F)$$

where $\nu_{F_{S,N}}$ is the normal bundle of $F_{S,N}$ in $M_{S,N}$.

2.6 A Normal Characteristic Class

Let G be a compact, connected Lie group. Suppose M is a closed almost complex G-manifold with an invariant almost complex structure. Let

$$\begin{array}{ccc} M & \xrightarrow{i_N} & M_{G,N} \\ & & \big\downarrow \pi_N \\ & & BG_N \end{array}$$

be an N-approximation to the Borel construction. The almost complex structures on T_*M and T_*BG_N together with G-invariance yields an almost complex structure on $T_*M_{G,N}$, so we have Chern classes

$$c_i(M_{G,N}) \in H^{2i}_{G,N}(M).$$

Definition 2.6.1. Let

$$c_i(M_G) \in H_G^*(M)$$

be the inverse limit class of $c_i(M_{G,N})$ given by (2.1.1). $c_i(M_G)$ is called the i^{th} *equivariant Chern class* of M.

The total Chern class $c(M_{G,N})$ restricts to $c(M)$ under the inclusion of the fibre,

$$\begin{aligned} i_N^*(c(M_{G,N})) &= c(M) \cdot c(\nu_M) \\ &= c(M) \end{aligned} \tag{2.6.1}$$

where ν_M is the normal bundle of M in $M_{G,N}$, since $c(\nu_M) = 1$ by local triviality of π_N. Since

$$i_N^* \pi_N^*(c(BG_N)) = 1, \tag{2.6.2}$$

$\pi_N^*(c(BG_N))$ is invertible in $H_{G,N}^*(M)$, so the class

$$\eta_N = \frac{c(M_{G,N})}{\pi_N^*(c(BG_N))} \in H_{G,N}^*(M)$$

also restricts to $c(M)$ under the inclusion of the fibre. Restricting η_N to the fixed point set F under the inclusion $i_{F,N} : F_{G,N} \to M_{G,N}$, we obtain

$$i_{F,N}^*(\eta_N) = c(F) \cdot c(\nu_{F_{G,N}}) \in H_{G,N}^*(F) \tag{2.6.3}$$

where $\nu_{F_{G,N}}$ is the normal bundle of $F_{G,N}$ in $M_{G,N}$. Since equations (2.6.1), (2.6.2) and (2.6.3) are independent of N for large N, we have

Theorem 2.6.2. *Let M be a closed almost complex manifold, G be a compact, connected Lie group preserving the almost complex structure, and F be the fixed point set* *of the action. The class* $\pi^*(c(BG))$ *is invertible in* $H_G^*(M)$ *so we may define the class*

$$\eta = \frac{c(M_G)}{\pi^*(c(BG))} \in H_G^*(M) \tag{2.6.4}$$

which satisfies

$$i^*(\eta) = c(M) \tag{2.6.5}$$

and which under the inclusion $i_F : F \to M$ *satisfies*

$$i_F^*(\eta) = c(F) \cdot c(\nu_{F_G}) \in H_G^*(F). \tag{2.6.6}$$

In the sequel, we will denote $\frac{c(M_G)}{\pi^*(c(BG))}$ by $c(M_G)/c(BG)$.

Remark 2.6.3. Let (ω, g, J) be a compatible triple with ω and g invariant. Then J is invariant. Thus, Theorem 2.6.2 applies to symplectic G-actions.

Chapter 3

McDuff-Type Constructions

3.1 Symplectic Reduction

Let (M, ω) be a Hamiltonian S^1-manifold with moment map μ. Assume that the action is semifree and that 0 is a regular value of μ. The level set $\mu^{-1}(0)$ is a codimension 1 submanifold of M and thus is coisotropic. See Definition 1.1.4. Ω decends to a symplectic structure on W/W^{Ω} and

$$\dim W/W^{\Omega} = \dim W - \operatorname{codim} W.$$

It turns out that the symplectic complement of the tangent space of $\mu^{-1}(0)$ is spanned by X_{S^1}, the fundamental vector field of the action, so the orbit space $\mu^{-1}(0)/S^1$ is a symplectic manifold called the *symplectic reduction* of M at 0. If the action is almost free, then the orbit space is an orbifold and one can still consider reduction. For actions of general compact Lie groups, one can show that, similar to the circle case, the level set is coisotropic and the symplectic complement of its tangent space is spanned by the fundamental vector fields of the action.

3.2 The Duistermaat-Heckman Theorem

Let (M, ω) be a symplectic manifold with a semifree Hamiltonian S^1-action. Suppose 0 is a regular value of the moment map μ. Since the critical values are isolated, we can find an $\epsilon > 0$ such that there are no critical values in the interval $(-\epsilon, \epsilon)$. The Duistermaat-Heckman theorem [DH82] relates the cohomology class of the reduced symplectic form on $M_0 = \mu^{-1}(0)/S^1$ with the class of the reduced form on $M_\delta = \mu^{-1}(\delta)/S^1$ for some $0 < |\delta| < |\epsilon|$. Specifically,

$$[\omega_\delta] = [\omega_0] + \delta \cdot u$$

where u is the Chern class of the S^1-bundle $\mu^{-1}(0) \to M_0$.

One approach to the proof utilizes the equivariant coisotropic embedding theorem to get an equivariant symplectomorphism between $\mu^{-1}(-\epsilon, \epsilon)$ and $\mu^{-1}(0) \times (-\epsilon, \epsilon)$ with symplectic form

$$\sigma = \pi^* \omega_0 - d(t\alpha)$$

where $\pi : \mu^{-1}(0) \to M_0$ is the canonical projection, α is a connection form on $\mu^{-1}(0)$ and t is a coordinate on $(-\epsilon, \epsilon)$. See Cannas da Silva [Ca01] for details. The coisotropic embedding theorem is originally due to Gotay [Got82] and can be thought of as a symplectic result since it relies on the Moser trick. There is another proof given by Audin [Au04] which we will outline here that uses only equivariant cohomology.

The map $j : \mu^{-1}(\epsilon) \to M$ is S^1-equivariant so induces a map

$$j \times_{S^1} 1 : \mu^{-1}(\epsilon) \times_{S^1} ES^1 \to M \times_{S^1} ES^1$$

and a map of Weil models

$$j_{\mathfrak{h}}^* : \Omega_{\mathfrak{h}}^*(M) \to \Omega_{\mathfrak{h}}^*(\mu^{-1}(\epsilon)).$$

Since the action is Hamiltonian, there is a closed, equivariant extension of ω in $\Omega_{\mathfrak{h}}^2(M)$ given by $\tilde{\omega} = \omega \otimes 1 + \mathcal{D}(\mu \otimes \theta)$ (2.3.3) where θ is a connection 1-form for $ES^1 \to BS^1$ and μ is the moment map. Since $j_{\mathfrak{h}}^*(d\mu \otimes \theta) = 0$,

$$j_{\mathfrak{h}}^*(\tilde{\omega}) = \omega \otimes 1 + \mu \otimes D\theta,$$

so under the isomorphism $\Omega_{\mathfrak{h}}^*(\mu^{-1}(\epsilon)) \simeq \Omega^*(M_{red})$, we get

$$\omega \otimes 1 + \mu \otimes D\theta \mapsto \omega_{red} + \epsilon \cdot \xi$$

where ξ is the Euler class of the S^1-bundle $\mu^{-1}(\epsilon) \to M_{red}$.

3.3 Symplectic Cuts and Symplectic Sums

The operation of *symplectic cutting* was introduced by Lerman [Le95]. Let (M, ω) be a semifree Hamiltonian S^1-manifold with moment map μ. Suppose that 0 is a regular value of μ, so the level set $\mu^{-1}(0)$ is a an S^1-bundle over M_0, the reduction at 0. The manifolds $M_{\mu \leq 0} = \mu^{-1}(-\infty, 0]$ and $M_{\mu \geq 0} = \mu^{-1}[0, \infty)$ have boundary $\mu^{-1}(0)$. Taking the quotient of $M_{\mu \leq 0}$, respectively $M_{\mu \geq 0}$, by the S^1-action on the boundary yields a closed manifold $\tilde{M}_{\mu \leq 0}$, respectively $\tilde{M}_{\mu \geq 0}$. In fact, they are semifree Hamiltonian S^1-manifolds with maximum, respectively minimum, fixed point component M_0. Lerman's construction defines the symplectic form on each half.

The inverse of symplectic cutting is known as *symplectic sum* and is due to Gompf [Gom95] and McCarthy-Wolfson [MW94]. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds with codimension 2 symplectic submanifolds $N_1 \approx N_2$. Suppose furthermore that the Euler classes of the normal bundles ν_{N_1} and ν_{N_2} satisfy $e(\nu_{N_2}) = -e(\nu_{N_1})$. Then we can glue M_1 to M_2 along the complements of the zero sections in ν_{N_1} and ν_{N_2} in such a way that when restricted to fibres is a sympletic inversion of the 2-dimensional annulus. This construction will not work in higher codimensions since there is no symplectic inversion of a $2k$-dimensional annulus for $k > 1$, since such a map would yield a symplectic form on S^{2k}.

3.4 The McDuff Construction

Let (M, ω) be a semifree non-Hamiltonian symplectic S^1-manifold and let X denote the fundamental vector field of the action. $H^1(M; \mathbb{Z}) \simeq [M, S^1]$, so if $[i_X\omega]$ is integral, then there is a smooth map $\mu : M \to S^1$ with $\mu^*(d\theta) = i_X\omega$ where $[d\theta]$ generates $H^1(S^1; \mathbb{Z})$. Since a semifree action is symplectic iff $L_X\omega = 0$, we can assume that ω is integral and thus, that $i_X\omega$ is integral. μ is called an S^1*-valued moment map* and was introduced by McDuff [Mc88].

Remark 3.4.1. A version of symplectic cutting can be accomplished on a manifold with a non-Hamiltonian action and S^1-valued moment map which will result in a Hamiltonian action. The action is lifted to a covering and ordinary symplectic cutting is done at two regular values covering the same value in S^1. Details on lifting the action can be found in Ortega-Ratiu [OR05].

Definition 3.4.2. For a Hamiltonian S^1-manifold, call the fixed point components not corresponding to the extrema of the moment map *internal*.

Suppose (M^{2n}, ω) is a semifree Hamiltonian S^1-manifold with internal fixed point components Z of codimension 4 and hence index 2. By McDuff [Mc88], the symplectic reductions at any regular value of the moment map μ are all diffeomorphic. Let $\tilde{M}$ denote their common diffeotype. Suppose c is a critical value of μ corresponding to a fixed point component Z. There is $\epsilon > 0$ such that $(c - \epsilon, c + \epsilon)$ contains no other critical values. Let $0 < \delta < \epsilon$. Also by McDuff [Mc88],

$$u_{c+\delta} - u_{c-\delta} = d_Z \tag{3.4.1}$$

where $u_{c+\delta}$, respectively $u_{c-\delta}$, is the first Chern class of the S^1-bundle $\mu^{-1}(c+\delta) \to \tilde{M}$, respectively $\mu^{-1}(c-\delta) \to \tilde{M}$, and d_Z is the Poincaré dual of Z considered as a submanifold of $\tilde{M}$. Note that this is a purely cohomological condition. If M results from appropriately cutting the universal cover of some non-Hamiltonian action with codimension 4 fixed point set F, then we must have

$$\sum_{Z \subseteq F} d_Z = 0 \tag{3.4.2}$$

as Z runs over the components of F.

Remark 3.4.3. Let $H^2_+(\tilde{M})$ be the positive cone in $H^2(\tilde{M})$, namely, the subspace on which $\int_{\tilde{M}} \omega_{red}^{n-1} > 0$. By nondegeneracy, the variation of the reduced sympletic form must take place entirely in $H^2_+(\tilde{M})$. This fact, coupled with the Duistermaat-Heckman theorem, implies that any semifree non-Hamiltonian symplectic S^1-action with codimension 4 fixed point set results in a closed, piecewise linear curve in $H^2_+(\tilde{M})$.

Remark 3.4.4. By McDuff [Mc88], given a fixed point component Z corresponding to a critical value c of μ, there is an $\epsilon > 0$ such that Z must be a symplectic submanifold of $\tilde{M}$ for all reduced symplectic forms $\omega_{c+\delta}$ for $|\delta| < \epsilon$. The condition that a fixed point component is symplectic implies

$$\int_Z \omega_{red}^{n-2} = \int_{M_{red}} d_Z \cdot \omega_{red}^{n-2} > 0$$

by Poincaré duality. Since

$$\int_{M_{red}} \left(\sum_{Z \subseteq F} d_Z \right) \cdot \omega_{red}^{n-2} = 0,$$

no fixed point component Z will be symplectic for all reduced forms ω_{red}. In fact, given any $\xi \in H^2(M_{red})$,

$$\int_{M_{red}} \left(\sum_{Z \subseteq F} d_Z \right) \cdot \xi = \sum_{Z \subseteq F} \int_Z \xi|_Z = 0.$$

Example 3.4.5 (McDuff [Mc88]). Let $B = T^4$ with coordinates (x^1, x^2, x^3, x^4), $L_{ij} = T^2 \subset B$ with coordinates (x^i, x^j) and $\sigma_{ij} = dx^i dx^j$. B is the diffeotype of M_{red} and the L_{ij} are the internal fixed point components. Let c_s be the Chern class of the S^1-bundle $\mu^{-1}(s) \to B$ for a regular value s, Z be a fixed point component, and d_Z be the Poincaré dual of Z in B. The original data for the McDuff construction is a family of symplectic forms τ_s on B.

$s \in [0,1)$	$\tau_s = K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + 2\sigma_{42}$	$c_s = 0$
$s = 1$	$Z = L_{13}$	$d_Z = [\sigma_{42}]$
$s \in (1,2)$	$\tau_s = K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + (3-s)\sigma_{42}$	$c_s = -[\sigma_{42}]$
$s = 2$	$Z = L_{42}$	$d_Z = [\sigma_{13}]$
$s \in (2,5)$	$\tau_s = K\sigma_{12} + K\sigma_{34} + (4-s)\sigma_{13} + (3-s)\sigma_{42}$	$c_s = -[\sigma_{13} + \sigma_{42}]$
$s = 5$	$Z = L_{31}$	$d_Z = -[\sigma_{42}]$
$s \in (5,6)$	$\tau_s = K\sigma_{12} + K\sigma_{34} + (4-s)\sigma_{13} - 2\sigma_{42}$	$c_s = -[\sigma_{13}]$
$s = 6$	$Z = L_{24}$	$d_Z = -[\sigma_{13}]$
$s \in (6,7]$	$\tau_s = K\sigma_{12} + K\sigma_{34} - 2\sigma_{13} - 2\sigma_{42}$	$c_s = 0$

We can double the McDuff data to obtain a closed, piecewise linear curve in the space of closed 2-forms on B.

$s \in [7,8)$	$\tau_s = K\sigma_{12} + K\sigma_{34} - 2\sigma_{13} - 2\sigma_{42}$	$c_s = 0$
$s = 8$	$Z = L_{31}$	$d_Z = -[\sigma_{42}]$
$s \in (8,9)$	$\tau_s = K\sigma_{12} + K\sigma_{34} - 2\sigma_{13} - (10-s)\sigma_{42}$	$c_s = [\sigma_{42}]$
$s = 9$	$Z = L_{24}$	$d_Z = -[\sigma_{13}]$
$s \in (9,12)$	$\tau_s = K\sigma_{12} + K\sigma_{34} - (11-s)\sigma_{13} - (10-s)\sigma_{42}$	$c_s = [\sigma_{13} + \sigma_{42}]$
$s = 12$	$Z = L_{13}$	$d_Z = [\sigma_{42}]$
$s \in (12,13)$	$\tau_s = K\sigma_{12} + K\sigma_{34} - (11-s)\sigma_{13} + 2\sigma_{42}$	$c_s = [\sigma_{13}]$
$s = 13$	$Z = L_{42}$	$d_Z = [\sigma_{13}]$
$s \in (13,14]$	$\tau_s = K\sigma_{12} + K\sigma_{34} + 2\sigma_{13} + 2\sigma_{42}$	$c_s = 0$

It remains to find a compatible semifree Hamiltonian S^1-manifold (M^6, ω) with moment map $\mu : M \to [0,14]$. That this can be done for codimension 4 fixed point components is due to Guillemin-Sternberg [GS89] and is detailed in Gonzalez [Gon05]. A semifree non-Hamiltonian symplectic S^1-manifold results from identifying the boundary components $\mu^{-1}(0)$ and $\mu^{-1}(14)$.

Remark 3.4.6. The McDuff data can be thought of as the cohomological part of the construction. The symplectic part, that is to say, the part which utilizes some variant of the Moser trick, lies in the actual construction of the semifree Hamiltonian S^1-manifold (M^6, ω). In fact, over an interval I of regular values of the moment map μ, McDuff [Mc88] shows that ω is determined, up to S^1-equivariant symplectomorphism which leaves the level sets of μ invariant, by a family of reduced forms τ_s where s varies over I, and the proof of this uses the Moser trick.

3.5 A Result on Almost Complex S^1-Actions

We now state and prove our main technical result which gives a cohomological constraint on McDuff-type constructions in higher dimensions. Given a semifree S^1-action on an almost complex manifold M which preserves the almost complex structure and has fixed point set F, recall that there is an equivariant splitting of the normal bundle $\nu = \nu^+ \oplus \nu^-$ of F into (± 1)-eigenbundles of the S^1-representation.

Theorem 3.5.1. *Let M^{2n} be a closed almost complex manifold with a semifree S^1-action preserving the almost complex structure. Suppose that the fixed point components F_i^{2k} are of codimension 4 and index 2. Let $c_i^{\pm}$ be the first ordinary Chern class of the (± 1)-eigenbundle $\nu_i^{\pm}$ of the normal bundle ν_i of the fixed point component F_i, $c_j(F_i)$ be the j^{th} ordinary Chern class of F_i for $0 \leq j \leq k = n-2$, and $[F_i]$ be the fundamental class of F_i. Then we have*

$$\sum_i \left(c_k(F_i) + c_{k-1}(F_i)(c_i^+ + c_i^-)\right)[F_i] = 0, \tag{3.5.1}$$

$$\sum_i \left(c_{k-1}(F_i)(c_i^+ - c_i^-) + c_{k-2}(F_i)((c_i^+)^2 - (c_i^-)^2)\right)[F_i] = 0, \tag{3.5.2}$$

and

$$\sum_i \left(c_{k-2}(F_i)((c_i^+)^2 - c_i^+ c_i^- + (c_i^-)^2) + c_{k-3}(F_i)((c_i^+)^3 + (c_i^-)^3)\right)[F_i] = 0. \tag{3.5.3}$$

Proof. Let $c(F_i)$ be the ordinary total Chern class of F_i and $c(\nu_i)$ be the equivariant total Chern class of ν_i. Since F_i is of codimension 4 and index 2, for $\eta = c(M_{S^1})/c(BS^1)$ (2.6.4) we have

$$\begin{aligned}\eta|_{F_i} &= c(F_i) \cdot c(\nu_i) \\ &= c(F_i)(1+t+c_i^+)(1-t+c_i^-) \\ &= c(F_i)(1+(c_i^+ + c_i^-)+e(\nu_i))\end{aligned}$$

where

$$e(\nu_i) = (t+c_i^+)(-t+c_i^-) = -t^2 + (c_i^- - c_i^+)t + c_i^+ c_i^-.$$

Localization (2.4.1) gives

$$\begin{aligned}\chi(M) &= \int_M c(M) \\ &= \int_M \eta|_M \\ &= \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)} \\ &= \sum_i \chi(F_i) + \sum_i \int_{F_i} \frac{c(F_i)(1+(c_i^+ + c_i^-))}{e(\nu_i)}.\end{aligned}$$

Furthermore,

$$\frac{1}{e(\nu_i)} = -\frac{1}{t^2}\left(1 + \left(\frac{c_i^- - c_i^+}{t} + \frac{c_i^+ c_i^-}{t^2}\right) + \cdots + \left(\frac{c_i^- - c_i^+}{t} + \frac{c_i^+ c_i^-}{t^2}\right)^k\right)$$

and

$$\chi(M) = \sum_i \chi(F_i),$$

so we get

$$\sum_i \int_{F_i} c(F_i)(1+(c_i^+ + c_i^-)) \left(1 + \left(\frac{c_i^- - c_i^+}{t} + \frac{c_i^+ c_i^-}{t^2}\right) + \cdots + \left(\frac{c_i^- - c_i^+}{t} + \frac{c_i^+ c_i^-}{t^2}\right)^k\right)$$
$$= 0 \in \mathbb{Q}(t).$$

The equations (3.5.1), (3.5.2) and (3.5.3) follow from vanishing of the constant part, the coefficient of $\frac{1}{t}$ and the coefficient of $\frac{1}{t^2}$, respectively. □

Remark 3.5.2. We only have to consider the coefficients of $\frac{1}{t^l}$ for $l \leq k$ since higher powers of $\frac{1}{t}$ have coeffcients of degree greater than $2k$. In general, we obtain the relation

$$\sum_i \left(c_{k-l}(F_i) \left(\frac{(c_i^+)^{l+1} - (-c_i^-)^{l+1}}{c_i^+ + c_i^-} \right) + c_{k-l-1}(F_i)((c_i^+)^{l+1} - (-c_i^-)^{l+1}) \right) [F_i] = 0.$$

3.6 S^1-Actions on 6-Manifolds with Fixed Surfaces

Let (M^6, ω) be a closed semifree non-Hamiltonian symplectic S^1-manifold with fixed point components F_i of dimension 2. We can use symplectic cutting on the universal cover $(\tilde{M}^6, \tilde{\omega})$, and by approprite scaling of $\tilde{\omega}$, we can make the lifted form integral and thus make all the critical values of the moment map on the cut manifold integral. We will, by abuse of notation, denote the closed semifree Hamiltonian S^1-manifold thus constructed as (M, ω), the corresponding moment map as μ and the internal fixed point components as F_i. The fixed point components F_{min} and F_{max} corresponding to the extreme values of μ are diffeomorphic to the symplectic reduction M_{red} at any regular value of μ since the internal components are of codimension 4 and hence of index 2. See McDuff [Mc88]. Furthermore, the Chern class c_{min} of the S^1-bundle $\mu^{-1}(min + \epsilon) \to M_{red}$ must be the same as the Chern class c_{max} of the S^1-bundle

$\mu^{-1}(max - \epsilon) \to M_{red}$. The equivariant Euler classes of the normal bundles ν_{min} of F_{min} and ν_{max} of F_{max} are

$$e(\nu_{min}) = t + c,$$
$$e(\nu_{max}) = -e(\nu_{min})$$

where t is a generator of $H^*(BS^1)$ and $c = c_{min} = c_{max}$. Denote the fundamental class of F_i by $[F_i]$ and let

$$c_i^{\pm} = c_1(\nu_i^{\pm})[F_i]$$

be the first Chern numbers of the corresponding line bundles. The equivariant Euler class of ν_i is

$$e(\nu_i) = (t + c_i^+ u_i)(-t + c_i^- u_i)$$
$$= -t^2 + (c_i^- - c_i^+)tu_i$$

where $u_i \in H^2(F_i)$ satisfies $u_i[F_i] = 1$. The total Chern class of F_i is

$$c(F_i) = 1 + \chi(F_i)u_i.$$

For $\eta = c(M_{S^1})/c(BS^1)$ (2.6.4) we have

$$\eta|_{F_{min}} = c(F_{min})(1 + e(\nu_{min})),$$
$$\eta|_{F_{max}} = c(F_{max})(1 + e(\nu_{max})),$$
$$\eta|_{F_i} = c(F_i)(1 + (c_i^+ + c_i^-)u_i + e(\nu_i)),$$

so localization (2.4.1) gives

$$\begin{aligned}
\chi(M) &= \int_M c(M) \\
&= \int_M \eta|_M \\
&= \int_{F_{min}} \frac{\eta|_{F_{min}}}{e(\nu_{min})} + \int_{F_{max}} \frac{\eta|_{F_{max}}}{e(\nu_{max})} + \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)} \\
&= \int_{F_{min}} \frac{c(F_{min})(1+e(\nu_{min}))}{e(\nu_{min})} + \int_{F_{max}} \frac{c(F_{max})(1+e(\nu_{max}))}{e(\nu_{max})} + \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)} \\
&= \chi(F_{min}) + \chi(F_{max}) + \int_{F_{min}} \frac{c(F_{min})}{e(\nu_{min})} + \int_{F_{max}} \frac{c(F_{max})}{e(\nu_{max})} + \sum_i \int_{F_i} \frac{\eta|_{F_i}}{e(\nu_i)} \\
&= \chi(F_{min}) + \chi(F_{max}) + \sum_i \chi(F_i) + \sum_i \int_{F_i} \frac{c(F_i)(1+(c_i^+ + c_i^-)u_i)}{e(\nu_i)}
\end{aligned}$$

and so

$$\sum_i \int_{F_i} \frac{c(F_i)(1+(c_i^+ + c_i^-)u_i)}{e(\nu_i)} = 0$$

since

$$\chi(M) = \chi(F_{min}) + \chi(F_{max}) + \sum_i \chi(F_i).$$

We have basically duplicated the method of proof of Theorem 3.5.1 in the case of a 6-manifold and thus have analogues of (3.5.1) and (3.5.2).

Remark 3.6.1. The preceeding calculations are not identical to those of the proof of Theorem 3.5.1 since F_{min} and F_{max} are of codimension 2 in M. We have used the non-Hamiltonian version of symplectic cutting which is briefly described in Remark 3.4.1, however, localization at η yields the same result for the uncut non-Hamiltonian manifold since

$$\chi(M_{cut}) = \chi(M_{uncut}) + \chi(F_{min}) + \chi(F_{max}).$$

Proposition 3.6.2. *Let M^6 be a closed semifree non-Hamiltonian symplectic S^1-manifold with fixed point set F consisting of surfaces F_i. Let $c_i^{\pm}$ be the first Chern* *numbers of the* (± 1)*-eigenbundles of the* S^1*-representation on the normal bundle* ν_i *of* F_i*. Then*

$$\sum_i \left(c_i^+ + c_i^- + \chi(F_i)\right) = 0 \tag{3.6.1}$$

and

$$\sum_i (c_i^+ - c_i^-) = 0. \tag{3.6.2}$$

We can obtain a sharper result than (3.6.1) when $c_1(M_{red}) = 0$. Since M_{red} is 4-dimensional, another result of McDuff [Mc88] says that the normal bundle of F_i in M_{red} is

$$\nu_i^{red} = \nu_i^+ \otimes \nu_i^-.$$

The Poincaré dual d_i of F_i in M_{red} is the Thom class of the normal bundle ν_i^{red} and thus satisfies

$$\begin{aligned} d_i|_{F_i} &= e(\nu_i^{red}) \\ &= c_1(\nu_i^+ \otimes \nu_i^-) \\ &= (c_i^+ + c_i^-)u_i. \end{aligned} \tag{3.6.3}$$

See Bott-Tu [BT82] for details on the Poincaré dual of a closed submanifold. An embedded j*-holomorphic* curve C in an almost complex 4-manifold X must satisfy the *adjunction formula*

$$C \cdot C - \langle c_1(X), C\rangle + 2 = 2g \tag{3.6.4}$$

where $C \cdot C$ is the *self-intersection* of C, $\langle\ ,\ \rangle$ is the pairing between cohomology and homology, and g is the genus of C. See McDuff-Salamon [MS04]. The formula is established by observing that

$$\begin{aligned} C \cdot C &= \int_X d_C \cdot d_C \\ &= \int_C e(\nu_C) \end{aligned} \tag{3.6.5}$$

and

$$\begin{aligned} \langle c_1(X), C \rangle &= \int_X c_1(X) \cdot d_C \\ &= \int_C e(C) + e(\nu_C) \end{aligned} \tag{3.6.6}$$

where d_C is the Poincaré dual of C in X and ν_C is the normal bundle of C in X.

Theorem 3.6.3. *Let M^6 be a closed semifree non-Hamiltonian symplectic S^1-manifold with fixed point set F consisting of surfaces F_i. Let $c_i^{\pm}$ be the first Chern numbers of the (± 1)-eigenbundles of the S^1-representation on the normal bundle ν_i of F_i. If $c_1(M_{red}) = 0$, then for each fixed point component F_i, we have*

$$c_i^+ + c_i^- + \chi(F_i) = 0. \tag{3.6.7}$$

Proof. Each F_i is a symplectic submanifold, and hence j-holomorphic curve, of M_{red} for an appropriate reduced form ω_{red}. See Remark 3.4.4. By (3.6.3) and (3.6.5),

$$F_i \cdot F_i = c_i^+ + c_i^-.$$

Since $\chi(F_i) = 2 - 2g$ where g is the genus of F_i, the result follows from the adjunction formula (3.6.4). □

Remark 3.6.4. $\sum_i d_i = 0$ by (3.4.2), so (3.6.6) gives another proof of (3.6.1).

We obtain additional results concerning the self-intersections of the fixed surfaces F_i thought of as submanifolds of a fixed copy of M_{red}. Recall that the F_i cannot all be symplectic by Remark 3.4.4.

Proposition 3.6.5. *Let M^6 be a closed semifree non-Hamiltonian symplectic S^1-manifold with fixed surfaces F_i. Then*

$$\sum_i \chi(F_i) = 2\sum_{i<j}(F_i \cdot F_j) \tag{3.6.8}$$

when the F_i are thought of as submanifolds of M_{red}. In particular, when $F_i \approx S^2$,

$$\sum_{i<j}(F_i \cdot F_j) = \#\text{ of fixed 2-spheres.} \tag{3.6.9}$$

Proof. $\sum_i d_i = 0$ by (3.4.2), so by (3.6.3) and (3.6.5), we have

$$\begin{aligned}\int_{M_{red}} \left(\sum_i d_i\right)^2 &= \sum_i (F_i \cdot F_i) + 2\sum_{i<j}(F_i \cdot F_j) \\ &= \sum_i (c_i^+ + c_i^-) + 2\sum_{i<j}(F_i \cdot F_j) \\ &= 0.\end{aligned}$$

The result follows from (3.6.1). □

Remark 3.6.6. The proofs of the results of this section are similar in spirit to those of Li [Li03b].

3.7 S^1-Actions with Isolated Fixed Points

The following Tolman-Weitsman result [TW00] on semifree S^1-actions on M^{2n} with isolated fixed points is recovered readily by localizing at $c(M_{S^1})/c(BS^1)$.

Theorem 3.7.1 (Tolman-Weitsman). *Let M^{2n} be a connected semifree symplectic S^1-manifold. Suppose the action has isolated fixed points. Then the action is Hamiltonian and the number N_k of fixed points of index $2k$ satisfies $N_k = \binom{n}{k}$, whence $\chi(M) = 2^n$.*

Proof. Let $\eta = c(M_{S^1})/c(BS^1)$ (2.6.4). Then

$$\begin{aligned} \eta|_p &= c(p) \cdot c(\nu_p) \\ &= (1+t)^{n-k}(1-t)^k \end{aligned}$$

and

$$\begin{aligned} e(\nu_p) &= c_n(\nu_p) \\ &= (-1)^k t^n \end{aligned}$$

where t is a degree 2 generator of $H^*(BS^1)$. Localization (2.4.1) gives

$$\begin{aligned} \chi(M) &= \int_M c(M) \\ &= \int_M \eta|_M \\ &= \sum_{p\in F} \frac{\eta|_p}{e(\nu_p)} \\ &= \sum_{p\in F} \frac{(1+t)^{n-k}(1-t)^k}{(-1)^k t^n} \\ &= \frac{1}{t^n}\left(\sum_{k=0}^{n} (-1)^k N_k (1+t)^{n-k}(1-t)^k\right) \end{aligned}$$

Evaluation at $t = 1$ yields

$$\begin{aligned} 2^n N_0 &= \chi(M) \\ &= \# \text{ of fixed points.} \end{aligned}$$

Thus $N_0 \neq 0$, so the moment map has an absolute minimum and therefore the action is Hamiltonian. Furthermore, since the fixed point component of the minimum is connected, $N_0 = 1$, so

$$\chi(M) = 2^n.$$

Applying $\frac{d}{dt}\big|_{t=1}$ to the equation

$$\chi(M)t^n = \sum_{k=0}^{n} (-1)^k N_k (1+t)^{n-k}(1-t)^k$$

gives

$$n\chi(M) = 2^{n-1}N_1 + n2^{n-1}N_0$$

from which $N_1 = n$ follows. Continuing in this fashion, we get

$$N_k = \binom{n}{k}.$$

□

Chapter 4

Cohomological Conditions

4.1 A Todd Genus Condition

A remarkable result of Feldman [Fe01] relates the Todd genus $Td(M)$ of a symplectic manifold M to Hamiltonian S^1-actions on M. Assume throughout that M is closed and connected.

Theorem 4.1.1 (Feldman)**.** *Suppose a symplectic S^1-manifold M has isolated fixed points. Then $Td(M) = 1$ if the action is Hamiltonian and $Td(M) = 0$ if the action is non-Hamiltonian. Any action on a symplectic manifold with $Td(M) > 0$ is Hamiltonian.*

Feldman's result follows from the formula

$$\chi_y(M) = \sum_s (-y)^{d_s} \chi_y(M_s)$$

where M_s is a component of the fixed point set and d_s is the number of negative weights of the S^1-representation on the normal bundle of M_s, namely, half the index. $\chi_y(M)$ is the Hirzebruch χ_y-genus of M and $Td(M) = \chi_y(M)|_{y=0}$.

We observe that for a Hamiltonian action $Td(M) = Td(M_{min})$ where M_{min} is the fixed point component corresponding the minimum of the moment map. Reversing the S^1-action interchanges M_{min} and M_{max} so

$$\begin{aligned} Td(M) &= Td(M_{min}) \\ &= Td(M_{max}). \end{aligned} \tag{4.1.1}$$

Lemma 4.1.2. *Let M be a semifree Hamiltonian S^1-manifold. Then*

$$\begin{aligned} Td(M) &= Td(M_{min}) \\ &= Td(M_{max}) \\ &= Td(M_{red}). \end{aligned}$$

Proof. Apply symplectic cutting to any regular value of the moment map and use (4.1.1). □

This is similar to a result of Li [Li03a] which states that for a semifree Hamiltonian S^1-manifold,

$$\begin{aligned} \pi_1(M) &= \pi_1(M_{min}) \\ &= \pi_1(M_{max}) \\ &= \pi_1(M_{red}). \end{aligned}$$

Theorem 4.1.3. *Let M be a sympletic manifold, N be a closed symplectic submanifold, and $\tilde{M}$ be the blow up of M along N. Then*

$$Td(\tilde{M}) = Td(M).$$

Proof. By Guillemin-Sternberg [GS89], we can consider M as the reduction $X_{-\epsilon}$ at $-\epsilon < 0$ of a semifree Hamiltonian S^1-manifold X where 0 is a critical value of the moment map with corresponding fixed point component N of (co)index 2. The reduction X_ϵ at $\epsilon > 0$ is the blow up $\tilde{M}$ of M along N. The result follows from Proposition 4.1.1. □

Proposition 4.1.4. *Let M be a symplectic S^1-manifold with $c_1(M) = 0$. Then*

$$Td(M) = 0.$$

Proof. Recall that a manifold M has a spin^c-structure if the second Stiefel-Whitney class $w_2(M)$ is the mod 2 reduction of an integral class $c \in H^2(M)$. Every symplectic manifold M has a spin^c-structure. For any spin^c-manifold M,

$$td(M) = \hat{a}(M)e^{c_1(M)/2}$$

where $td(M)$ is the Todd class of M and $\hat{a}(M)$ is the Dirac $\hat{A}$-class of M. If $c_1(M) = 0$, then M is a spin S^1-manifold, so

$$Td(M) = \hat{A}(M) = 0$$

by a result of Atiyah-Hirzebruch [AH70]. □

Definition 4.1.5. A *Calabi-Yau* manifold is a Kähler manifold with $c_1 = 0$.

Corollary 4.1.6. *If M is a Calabi-Yau manifold equipped with a Hamiltonian S^1-action, then $Td(M_{min}) = Td(M_{max}) = 0$. In particular, M_{min} and M_{max} cannot be points.*

4.2 $\omega|_{\pi_2(M)} = 0$ and Related Conditions

Let (M, ω) be a sympletic manifold. Let ξ be the fundamental vector field of a symplectic S^1-action on M. Then $L_\xi \omega = 0$, so by the Cartan homotopy formula (1.3.1), $i_\xi \omega$ is closed. The action is Hamiltonian iff $i_\xi \omega$ is exact. Cohomologically, M is a Hamiltonian S^1-manifold iff ω has a closed equivariant extension in $H^*_{S^1}(M)$ iff $d_2 \omega = 0$ in the Serre spectral sequence of the Borel construction M_{S^1}. It is known that the Chern classes of M survive to E_∞ in the Serre spectral sequence of M_{S^1}, so an action on a *monotone* symplectic manifold, $k\omega = c_1$, must be Hamiltonian. Note that the existence of fixed points is not needed.

Hamiltonian S^1-actions must have fixed points since the image of the moment map will be a closed interval whose endpoints will be critical values. In fact, the moment map is a Morse-Bott function so the Serre spectral sequence of M_{S^1} collapses at E_2, namely, $d_2(\omega) = 0$ implies $E_2 = E_\infty$.

Proposition 4.2.1 (Ono [On92]). *Let (M, ω) be a semifree non-Hamiltonian symplectic S^1-manifold. If $\omega|_{\pi_2(M)} = 0$, then the fixed point set $F = \emptyset$.*

Proof. (Allday [Al06]) Suppose $F \neq \emptyset$. Then there is a nontrivial element $\alpha \in \pi_1(M - F)$ which is not in the image of $\pi_1(i_s)$ where $i_s : \mu^{-1}(s) \to M - F$ is the inclusion of the level set of any regular value s. Joining the image of a representative of α to a fixed point of the action by a path in $M - F$ yields $S^1 \vee [0, 1]$. Using the circle action, we obtain T^2 with a 2-disk sewed onto a meridian. The resulting space is homotopic to $S^2 \vee S^1$ and thus represents a nontrivial element $\hat{\alpha} \in \pi_2(M)$ satisfying $\omega(\hat{\alpha}) \neq 0$, a contradiction. □

Consider the condition $(k\omega - c_1(M))|_{\pi_2(M)} = 0$. M is called *spherically monotone.* Let $X = S^2 \times T^2$ and let S^1 act on the second factor of T^2 in the standard fashion.

By the long exact homotopy sequence of $S^2 \to X \to T^2$, $\pi_2(X) = \pi_2(S^2)$, so given an element $\alpha \in \pi_2(X)$, $\alpha = i_*(\hat{\alpha})$ where $\hat{\alpha} \in \pi_2(S^2)$ and $i : S^2 \to X$ is the inclusion into the first factor. Let $1 \in \pi_2(X)$ be $i_*([\mathrm{id}_{S^2}])$. Then $\alpha = n \cdot 1$ so

$$(k\omega - c_1(X))(\alpha) = kn\omega(1) - 2n = 0$$

for $k = \frac{2}{\omega(1)}$ which is independent of n. On the other hand, if S^1 acts on the first factor, then X has fixed points, so spherical monotonicity says nothing about the existence of fixed points. However, Allday [Al06] has shown that if M is spherically monotone and has a fixed point, then the action is Hamiltonian.

Remark 4.2.2. Another way of seeing that $X = S^2 \times T^2$ is spherically monotone is to note that $c_1(X) = c_1(S^2)$.

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