end-to-end average distortion with the SNR, which was referred to as the distortion SNR exponent $a^\star(\eta)$, for given spectral efficiency η. We first derived an upper bound on $a^\star(\eta)$ based on an informed transmitter that has instantaneous knowledge of the channel capacity. Then the exponent achievable with a separation based scheme was computed. Finally, we proposed HDA source–channel coding schemes that outperform the separated exponent for all η. Remarkably, the HDA scheme for $\eta > 2\min(M,N)$ was shown to achieve the optimal distortion exponent for general M and N. We also showed how to construct practical space–time coding schemes using diversity–multiplexing tradeoff optimal space–time codes and scalar quantization.

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A Note on the Optimality of Variant-I Permutation Modulation Codes

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***Abstract*—In this correspondence, the optimality of variant-I permutation codes initially proposed by Slepian is shown in a simple way.**

***Index Terms*—Group codes, permutation codes, permutation modulation.**

I. INTRODUCTION

In [1], two variants of permutation modulation (PM) codes are introduced. A variant-I code C is defined by l integers $m_1 \leq m_2 \leq \cdots \leq m_l$ satisfying $\sum_{i=1}^{l} m_i = n$ and the n-dimensional vector

$$\boldsymbol{x} = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_l, \ldots, x_l) \tag{1}$$

where for $i = 1, \ldots, l$, x_i appears m_i times. The codebook is formed of all

$$M = \frac{n!}{m_1! \quad m_2! \ldots m_l!} \tag{2}$$

possible distinct permutations of $\boldsymbol{x}$. The rate of the code is $R = \log_2 M/n$ and its minimum squared Euclidean distance (MSED) is defined as

$$d_{\min}^2 = \min_{\boldsymbol{x} \neq \boldsymbol{x}' \in C} \|\boldsymbol{x} - \boldsymbol{x}'\|^2. \tag{3}$$

Manuscript received January 18, 2006; revised January 22, 2007.

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Communicated by A. Ashikhmin, Associate Editor for Coding Theory.
Digital Object Identifier 10.1109/TIT.2007.901182

Based on the results of [2], the design of an optimum PM variant-I code can be formulated as follows: "Given the set $\{m_1, m_2, \ldots, m_l\}$, find the corresponding set of points $\{x_1, x_2, \ldots, x_l\}$ such that $d^2_{\min} = 2$ and $\|\boldsymbol{x}\|^2$ is minimized."[1] In fact, [2] provides the solution to this problem but for the last step of the proof, it refers to [3]. In the following, we provide an alternative answer to this part.

In can be noted that the optimum solution derived in [4] is more restrictive as the values m_i's need to satisfy $m_i = m_{l-i-1}$. As a result, many optimum codes found by our approach are not considered by that of [4]. On the other hand, it is straightforward to show that the solution to our optimization design problem for variant-II PM codes is equivalent to that of [4].

II. Optimum Variant-I Codes

In this section, the optimality of variant-I PM codes is derived based on the following.

Lemma 2.1: Consider l integers $m_1 \le m_2 \le \cdots \le m_l$ satisfying $\sum_{i=1}^{l} m_i = n$. Then

$$\frac{1}{n}\sum_{j=1}^{\lfloor l/2 \rfloor} j(m_{l-2j+1} - m_{l-2j}) \le 1/2 \tag{4}$$

with $m_0 = 0$

Proof: For simplicity of the notations, we assume l is odd, so that $l = 2a - 1$. The case l even follows in a similar way. Proving (4) by contradiction, we assume that

$$\sum_{j=1}^{a-1} j(m_{l-2j+1} - m_{l-2j}) > n/2. \tag{5}$$

Since

$$m_l + \sum_{j=1}^{a-1}(m_{l-2j+1} + m_{l-2j}) = n, \tag{6}$$

multiplying (5) by 2 and subtracting (6) from both sides yields, after shifting the pair grouping of consecutive values of m_i's by one

$$\sum_{j=1}^{a-1}(2j-1)(-m_{l-2j+2} + m_{l-2j+1}) - (2a-1)m_1 > 0 \tag{7}$$

which is a contradiction as all the terms are nonpositive.

A variant-I PM code with $d^2_{\min} = 2$ is optimum if and only if the three following conditions are satisfied (see, e.g., [2] for an equivalent formulation).

- Condition 1: As a set, $\{x_1, \ldots, x_l\} = \{b, b+1, \ldots, b+l-1\}$ for some real value b.
- Condition 2: $\sum_{i=1}^{l} m_i x_i = 0$.
- Condition 3: If $m_i \le m_j$, then $|x_i| \ge |x_j|$.

Condition 1 ensures $d^2_{\min} = 2$ while Condition 2 implies that although $\boldsymbol{x}$ has dimension n, the effective dimension of the code is $n-1$, so that the effective code rate becomes $R = \log_2 M/(n-1)$.

In the following, we present a simple way to achieve these three conditions and for simplicity of the notations, we again assume l is odd; the results are extended in a straightforward way to the case when l is even. For $l = 2a - 1$ and $i = 1, \ldots, l$, we assign to each m_i the value $a + k_i$, with $k_i = (-1)^{l-i}\lceil (l-i)/2 \rceil$. In other words, we assign a to m_l, $a-1$ to m_{l-1}, $a+1$ to $m_{l-2}, \ldots, 1$ to m_2, l to m_1. It follows that

$$\frac{1}{n}\sum_{i=1}^{l} m_i(a+k_i) = a - \frac{1}{n}\sum_{j=1}^{a-1} j(m_{l-2j+1} - m_{l-2j}) \tag{8}$$

with $m_{l-2j+1} - m_{l-2j} \ge 0$.

Defining $x_i = a + k_i - (1/n)\sum_{i=1}^{l} m_i(a+k_i)$, we have

$$x_i = k_i + \frac{1}{n}\sum_{j=1}^{a-1} j(m_{l-2j+1} - m_{l-2j}). \tag{9}$$

As a result, Conditions 1 and 2 are straightforwardly satisfied. Condition 3 follows from Lemma 2.1, which ensures that the monotonicity of the values $|k_i|$ is preserved.

Note that this result also validates the optimum construction presented in [5] which implicitly assumes that the value a_m has the same number of terms to its left and its right.

Finally, as indicated in Section I, optimum variant II-a and variant II-b codes are obtained in a straighforward way with for $i = 1, \ldots, l$, $x_i = l - i$ and $x_i = l - i + 1/\sqrt{2}$, respectively.

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[1] $d^2_{\min} = 2$ allows us to choose integer values for $x_i - x_{i-1}$'s [2].