Fourier Coefficients of Weak Harmonic Maass Forms of Integer Weight: Calculation, Rationality, and Congruences

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I. Acknowledgements

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II. Abstract

With the numerous discoveries published recently in the study of harmonic weak Maas forms and the closely associated weakly holomorphic modular forms, it is important to understand the arithmetic properties of these forms. This paper will focus on the holomorphic parts from the q-expansions of harmonic weak Maas forms since it has been recently discovered that they have numerous applications throughout number theory. Although the numerical data cannot be used to completely prove the irrationality of these coefficients, it does provide strong evidence to suggest that conclusion. Once this is established, we branch out into congruences that were observed through the investigation of various harmonic weak Maas forms. A recurring pattern led to the formulation of a new theorem to describe the behavior of the coefficients. It is important to note that although the numerical examples presented here are obtained focusing on modular forms of integral weight, the same techniques can be applied to produce similar results for those of half-integral weight as well. The reason for looking exclusively at integral weights is to simplify the calculations involved when finding the coefficients, thus cutting down on computation time as well as increasing the number of coefficients that can be used as numerical data. The proposed theorem summarizes the new findings and is based on congruences demonstrated through simulations. This new theorem is in the process of being proved by Pavel Guerzhoy and his research team.

1. Introduction.

Harmonic weak Maass forms have recently begun to play an important role in different areas of number theory. The first application found was connected to Ramanujan's mock theta-functions. These are certain q-series of combinatorial origin written down by Ramanujan in his last letter to Hardy in 1913. Since then, an amount of work was done by various mathematicians (cf. e. g. [4]) towards the understanding of transformation properties of these functions. This work was motivated primarily by the combinatorics implied by these properties. However, as it was realized by late 1990s, the identities (transformation laws) for mock theta-functions are related to a surprising collection of mathematical subjects such as Artin L-functions in number theory, hypergeometric functions, partitions, Lie theory, Mordell integrals, and modular forms. The most comprehensive analysis of these transformation properties was recently presented by Zwegers [5]. The contemporary point of view on mock theta functions is based on Zwegers discoveries: mock theta functions are holomorphic parts of weak harmonic Maass forms of weight $1/2$. In this connection, in recent years the theory of weak harmonic Maass forms has been developed in various directions. The q-expansion coefficients of their holomorphic parts is a subject of particular interest. Although classical mock theta-functions appear in the case of weight $1/2$, other weights, integral in particular, also received a consideration. The q-expansion coefficients of classical mock theta-functions are rational numbers by definition. These are, in a sense, exceptional series. More generally, it turns out that in the case of weight $1/2$ generic coefficients are transcendental numbers, and the algebraicity of a single coefficient of this q-expansion implies important consequences [2]. In the case of integral weight, the rationality, and, as a natural generalization, the algebraicity question remained unclear until recently. Striking results in this direction were obtained in [1]. It turns out that in certain cases all q-expansion coefficients of the holomorphic parts of weak harmonic Maass forms are provably algebraic numbers. The investigation of their arithmetic properties was initiated in this paper. Note that, at the same time, there are cases, when, as computations suggest, these coefficients are not rational numbers. A proof of such a statement would imply important long-standing conjectures (cf. [3]).

This thesis is devoted to further investigation of arithmetic properties of the q-expansion coefficients of holomorphic parts of weak harmonic Maass forms of integral weight. We introduce notations and present necessary background material in Chapter 2. A general theoretical result on the algebraicity of the q-expansion coefficients in question is also presented at the end of Chapter 2. Chapter 3 outlines the theory behind the numerical calculations seen throughout the rest of the paper. Chapter 4 is devoted to numerical examples which support this theoretical result mentioned in Chapter 2. These examples are obtained using the techniques of Poincaré series described in Chapter 3. In Chapter 5, we experiment further to investigate, in the spirit of [1], arithmetic properties of these q-expansion coefficients. We provide numerical evidence for certain congruences which are much stronger, and more general then those obtained in [1]. A brief discussion on the precision observed in all calculations and what it means on the overall results is presented as Chapter 6. The actual code used to calculate the various coefficients is attached as an appendix.

2. Background and notations.

Throughout suppose that the weight, $k \in \mathbb{N}$, and the level ,$N \in \mathbb{Z}^+$, is fixed. Let $M_k^!(\Gamma_0(N), \chi)$ denote the space of wight k weakly holomorphic modular forms on $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, c \equiv 0 \bmod N \right\}$ with Nebentypus χ. Recall that a *weakly holomorphic modular form* is any meromorphic modular form whose poles (if any) are supported at cusps. Weakly holomorphic modular forms naturally sit in spaces of harmonic weak Maass forms.

Here we recall definitions and facts about harmonic weak Maass forms. Throughout, let $z = x + iy \in \mathbb{H}$, the upper half of the complex plane, with $x, y \in \mathbb{R}$. Also, throughout suppose that the weight, $k \in \mathbb{N}$, and the level ,$N \in \mathbb{Z}^+$, is fixed. We define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

Suppose that χ is a Dirichlet character modulo N. Then a *harmonic weak Maas form of weight k* on $\Gamma_0(N)$ with Nebentypus χ is any smooth function on $\mathbb{H}$ satisfying:

(1) $f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$;
(2) $\Delta_k f = 0$;
(3) There is a polynomial $P_f = \sum_{n \leq 0} c_f^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_f = \mathcal{O}(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. Analogous conditions are required at all cusps.

The polynomial $P_f \in \mathbb{C}[q^{-1}]$ is called the *prinicpal part* of f at the corresponding cusp. We denote the vector space of these harmonic weak Maass forms by $H_k(\Gamma_0(N), \chi)$ and the vector space of elliptic cusp forms on $\Gamma_0(N)$ by $S_k(\Gamma_0(N), \bar{\chi})$.

It is natural to investigate the arithmeticity of the Fourier coefficients of such Maass forms. Define the differential operator

$$\xi = \xi_k = 2iy^k \overline{\frac{\partial}{\partial \bar{z}}}.$$

It is a fact that $\xi_{2-k} : H_{2-k}(\Gamma_0(N), \chi) \to S_k(\Gamma_0(N), \chi)$. It is not difficult to make this more precise using Fourier expansions. In particular, every harmonic weak Maass form $f(z)$ of weight $2-k$ has a Fourier expansion of the form

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi |n| y) q^n, \tag{1}$$

where $\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t}\, dt$ is the incomplete Gamma-function, $z = x + iy \in \mathbb{H}$, with $x, y \in \mathbb{R}$, and $q := e^{2\pi i z}$. A straighforward calculation shows that $\xi_{2-k}(f)$ has Fourier expansion

$$\xi_{2-k}(f) = -(4\pi)^{k-1} \sum \overline{c_f^-(-n)} n^{k-1} q^n.$$

As seen from its Fourier expansion (1), $f(z)$ naturally decomposes into two summands

$$f^+(z) := \sum_{n \gg -\infty} c_f^+(n) q^n,$$

$$f^{-}(z) := \sum_{n<0} c_f^{-}(n)\,\Gamma(k-1, 4\pi |n| y)\, q^n.$$

Therefore, $\xi_{2-k}(f)$ is given simply in terms of $f^{-}(z)$, the *non-holomorphic part of f*. It has also been shown that $f^{+}(z)$, the *holomorphic part of f*, is also intimately related to weakly holomorphic modular forms as proved in [1] (Theorem 1.1).

Recall the Maas raising and lowering operators (see [6]) R_k and L_k on functions $f : \mathbb{H} \to \mathbb{C}$ which are defined by

$$R_k = 2i\frac{\partial}{\partial z} + ky^{-1} = i\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + ky^{-1},$$
$$L_k = -2iy^2\frac{\partial}{\partial \bar{z}} = -iy^2\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

With respect to the Peterson slash operator, these operators satisfy the intertwining properties

$$R_k(f\ |_k\ \gamma) = (R_k f)\ |_{k+2}\ \gamma,$$
$$L_k(f\ |_k\ \gamma) = (L_k f)\ |_{k+2}\ \gamma,$$

for any $\gamma \in SL_2(\mathbb{R})$. The Laplacian Δ_k can be defined in terms of R_k and L_k by

$$-\Delta_k = L_{k+2}R_k + k = R_{k-2}L_k.$$

If f is an eigenfunction of Δ_k satisfying $\Delta_k f = \lambda f$, then

$$\Delta_{k+2}R_k f = (\lambda + k)\, R_k f,$$
$$\Delta_{k-2}L_k f = (\lambda - k + 2)\, L_k f.$$

For any positive integer n we put

$$R_k^n := R_{k+2(n-1)} \circ \ldots \circ R_{k+2} \circ R_k$$

We also let R_k^0 be the identity. The differential operator

$$D := \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$$

satisfies the relation

$$R_k = -4\pi D + k/y.$$

Let $g \in S_k(\Gamma_0(N))$ be a normalized newform with Fourier expansion

$$g = \sum_{n\geq 1} b(n)\, q^n.$$

Let $g^c(\tau) = \overline{g(-\bar{\tau})} = \sum \overline{b(n)} q^n$, where the bar denotes complex conjugation. Then $g^c \in S_k(\Gamma_0(N), \bar{\chi})$ is a newform. In the most interesting examples, χ is either quadratic or trivial. In these cases $g^c = g$. Denote $K = K_g$ to be the algebraic number field generated over $\mathbb{Q}$, the rationals, by adjoining all Fourier coefficients of g, and the roots of unity of order $N \cdot N_\chi$. Throughout the remainder of this paper, we assume explicitly that functions in question will have real coefficients and hence will drop the complex conjugation. For simplification, we will now simplify notation for the vector space of harmonic weak Maas forms of weight k as $H_k(\Gamma_0(N))$ and the vector space of elliptic cusp forms of weight k as $S_k(\Gamma_0(N))$.Following [1], we say that a harmonic weak Maass form $f \in H_{2k}(\Gamma_0(N))$ is *good for* g if it satisfies the following properties:

(1) The principal part of f at the cusp ∞ belongs to $F_g[q^{-1}]$.

(2) The principal parts of f at the other cusps of $\Gamma_0(N)$ are constant.
(3) We have that $\xi_{2-k}(f) = ||g||^{-2} g$.

The existence of a f which is good for g (for every g) is proved in [1, Proposition 5.1].
The following theorem is the main result of [1]

Theorem 1. *Let $g \in S_k(\Gamma_0(N))$ be a normalized newform with complex multiplication. If $f \in H_{2-k}(\Gamma_0(N))$ is good for g, then all coefficients of f^+ are in $F_g(\zeta_M)$, where $\zeta_M := e^{2\pi i/M}$, and $M = ND$ where D is the discriminant of the field of complex multiplication.*

We will not introduce the notion of CM-forms here. It suffices to say that this is a special tiny (although infinite) class of cusp forms. Our result is more general (although less precise). We prove that a certain correction of the holomorphic part of a harmonic weak Maass form always has algebraic q-expansion coefficients.

Theorem 2. *Let f be good for g. Let*

$$E_g = \sum_{n \geq 1} \frac{b(n)}{n^{k-1}}.$$

Then $\alpha - c_f(1) \in K$ implies

$$\mathcal{F}_\alpha := f^+ - \alpha E_g \in K(q).$$

As an immediate corollary, we conclude that the coefficients $c_f(n)$ are not too far from being algebraic numbers.

Corollary 1. *The transcendence degree of the field $\mathbb{Q}(c_f(n))$ over K is at most one.*

We now prove Theorem 2.

Proof of Theorem 2. This is merely a slight modification of an argument from the proof of [1, Theorem 1.3].

We use the Hecke action on f. Let $T(m)$ be the m-the Hecke operator for the group $\Gamma_0(N)$ and Nebenypus χ. Using the same argument as in [2, Lemma 7.4] (cf. [1], the proof of Theorem 1.3), we have that

$$f|_{2-k}T(m) = m^{k-1} b(m) f + R_m,$$

where $R_m \in M^!_{2-k}(\Gamma_0(N), \bar{\chi})$ is a weakly holomorphic modular form with coefficients in K. We apply the differential operator D^{k-1} to this identity and use the (obvious) commutation relation

$$D^{k-1}\left(H|_{2-k}T(m)\right) = m^{k-1}\left(D^{k-1}H\right)|_k T(m)$$

valid for any 1-periodic function H. We obtain

$$\left(D^{k-1}f\right)|_k T(m) = b(m)\left(D^{k-1}f\right) + m^{1-k} D^{k-1} R_m$$

Since $D^{k-1} f = D^{k-1} f^+$, and $D^{k-1} E_g = g$, we take into the account that $g|_k T(m) = b(m) g$, and conclude that

$$\left(D^{k-1}(f^+ - \alpha E_g)\right)|_k T(m) = b(m)\left(D^{k-1}(f^+ - \alpha E_g)\right) + m^{1-k} D^{k-1} R_m \qquad (2)$$

We choose $\alpha \in \mathbb{C}$ such that $c_f(1) - \alpha \in K$, and claim that the q-series in the left-hand side $D^{k-1}(f^+ - \alpha E_g) = D^{k-1}\mathcal{F}_\alpha$ has its coefficients in $c_\alpha(n) \in K$. Indeed, we make use of the formula for the action of Hecke operators on Fourier expansions, equate the coefficients of q^n in (2), and conclude that for any prime m

$$c_\alpha(mn) + \chi(m)m^{k-1}c_\alpha(n/m) - b(m)c_\alpha(n) \in K$$

An inductive argument then demonstrates the claim and finishes the proof. □

Note that the key concepts are being summarized for the purpose of this paper and more indepth discussion and explanations can be found in the references listed as well as other sources. Having laid the basic groundwork for this area, we now move on to the focus of this paper.

3. Method of numerical calculation. Poincaré series.

In order to conduct numerical experiments with holomorpic part of weak harmonic Maass forms, we need an independent method to construct these forms which provides explicit formulas for their Fourier coefficients. We do this with the help of Poincaré series. The functions $Q(m,k,N;z)$ (see the definition below) are weak harmonic Maass forms by construction. It is known that they are bounded at all cusps different from infinity. Thus in the case when $\dim S_k(N) = 1$ they are, up to a multiple, good for the unique cusp form.

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, define $j(A,z)$ by

$$j(A,z) := (cz+d)$$

As usual, for such A and functions $f : \mathbb{H} \to \mathbb{C}$, we let

$$(f \mid_k A)(z) := j(A,z)^{-k} f(Az).$$

Let m be an integer, and let $\varphi_m : \mathbb{R}^+ \to \mathbb{C}$ be a function which satisfies $\varphi_m(y) = O(y^\alpha)$, as $y \to 0$, for some $\alpha \in \mathbb{R}$. If $e(\alpha) := e^{2\pi i\alpha}$ as before, then let

$$\varphi_m^*(z) := \varphi_m(y)\, e(mx).$$

Such functions are fixed by the translations $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}$.

Given this data, for integers $N \geq 1$, we define the generic Poincaré series

$$\mathbb{P}(m,k,\varphi_m,N;z) := \sum_{A \in \Gamma_\infty \backslash \Gamma_0(N)} (\varphi_m^* \mid_k A)(z).$$

The series we want to work with is modular, and its Fourier expansion is given in terms of the I-Bessel function and the Kloosterman sums

$$K(m,n,c) := \sum_{v(c)^\times} e\left(\frac{m\overline{v} + nv}{c}\right).$$

Here v runs through the primitive residue classes modulo c, and $v\overline{v} \equiv 1 \pmod{c}$. Let $M_{v,\mu}(z)$ be the usual M-Whittaker function. For complex s, let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}}\, M_{\frac{k}{2}sgn(y), s-\frac{1}{2}}(|y|),$$

and for $m \geq 1$ let $\varphi_{-m}(z) := \mathcal{M}_{1-\frac{k}{2}}(-4\pi m y)$. For $k \in 2\mathbb{N}$ and integers $N \geq 1$, we let

$$Q(-m,k,N;z) := \frac{1}{(k-1)!} \cdot P\left(-m, 2-k, \varphi_{-m}, N; z\right).$$

It is established in [1, Proposition 6.2] that Q decomposes into a holomorphic part, Q^{+}, and a non-holomorphic part, Q^{-}. Since we interested in the holomorphic part, we note that it can be expressed as the q-series

$$Q^{+}(-m,k,N;z) = q^{-m} + \sum_{n=0}^{\infty} b(-m,k,N;n)\, q^{n},$$

where

$$b(-m,k,N;0) = -\frac{2^{k}\pi^{k}(-1)^{\frac{k}{2}} m^{k-1}}{(k-1)!} \cdot \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N}}} \frac{K(-m,0,c)}{c^{k}},$$

and where for positive integers n we have

$$b(-m,k,N;n) = -2\pi(-1)^{\frac{k}{2}} \cdot \sum_{\substack{c>0 \\ c\equiv 0 \pmod{N}}} \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \frac{K(-m,n,c)}{c} \cdot I_{k-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right).$$

Note that this is well known for $2 < k \in 2N$. The case for $k = 2$ follows from an argument of the analytic continuation in k with Fourier expansions. The following calculations were done using this representation.

4. Examples of rationality.

The following examples demonstrate how we can obtain rational coefficients through a minor adjustment.

Example (1). In the case when $g = \Delta$, the unique weight 12 cusp form on $SL_2(\mathbb{Z})$, the first q-expansion coefficients of f^{+} were calculated in [3]:

$$\begin{aligned} 39916800 f^{+} = \ & 39916800 q^{-1} - \tfrac{2615348736000}{691} \\ & - 73562460235.68364... q - 929026615019.11308... q^{2} \\ & - 8982427958440.32917... q^{3} - 71877619168847.70781... q^{4} - ..., \end{aligned}$$

and the coefficients $c_f(n)$ of positive powers of q are conjectured to be irrational numbers. We put

$$\alpha = c_f(1) = -\frac{73562460235.68364746919338632959387235627725962624646839904...}{39916800},$$

and increase the precision of calculations to find, in accordance with Theorem 2 a series with rational coefficients:

$$\begin{aligned} 39916800 \mathcal{F}_{\alpha} = \ & 39916800 q^{-1} - \tfrac{2615348736000}{691} - 929888675100 q^{2} - \tfrac{80840909811200}{9} q^{3} \\ & - \tfrac{4600169279088075}{64} q^{4} - \tfrac{194518227202453389312}{390625} q^{5} \\ & - \tfrac{9224840575997200}{3} q^{6} ... \end{aligned}$$

Example (2). In the case where $N=4, k=6$, let

$$g=q-12q^{3}+54q^{5}-88q^{7}-99q^{9}+540q^{11}-418q^{13}+...,$$

then first q-expansion coefficients of f^{+} are calculated as:

$$\begin{aligned}120f^{+}= \; & 120q^{-1}+45.949152056947679q-94.120945780590008q^{3}\\ & +185.114001347544055q^{5}-353.857531110906848q^{7}\\ & +648.409474063004660q^{9}-1145.608207697494882q^{11}+...,\end{aligned}$$

We put

$$\alpha=c_{f}\left(1\right)=\frac{45.9491520569476793215332447011055832\text{98...}}{120}$$

and increase precision of calculations to find rational coefficients of the modified series:

$$120\mathcal{F}_{\alpha}= \; 120q^{-1}-\tfrac{2480}{27}q^{3}+\tfrac{4608}{25}q^{5}-\tfrac{5943240}{16807}q^{7}+\tfrac{1418240}{2187}q^{9}-\tfrac{184526160}{161051}q^{11}+...$$

Note that by showing that the rationalization of coefficients is possible, we have shown that the original coefficients of f^{+} are are most likely irrational numbers to begin with. Although the numerical examples do not constitute a complete proof of this phenomena, they do provide us with strong evidence and allow us to accept the results as being highly probable, thus allowing us to study the q-expansions accordingly.

5. Examples of congruences.

Taking the previous results into account, it now becomes viable to consider other arithmetic properties of the coefficients of the q-expansions from weak harmonic Maass forms. Theorem 2 makes it natural to consider the arithmetic properties of the series $\mathcal{F}_{\alpha}$, which is a slightly modified holomorphic part f^{+} of the harmonic weak Maas form f. Let p be a prime. We fix, once and for all, an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ and an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$. This embedding, in particular, determines an extension of the p-adic valuation to K. We denote by $ord_p:\overline{\mathbb{Q}}\rightarrow\mathbb{Q}$ the p-adic order such that $ord_p\left(p\right)=1$. It will be clear from the results that the Fourier coefficients of $\mathcal{F}_{\alpha}$ may have unbounded negative p-adic order (cf. [1, Remark 5]). Let

$$F_{\alpha}:=D^{k-1}\mathcal{F}_{\alpha}=\sum_{n\gg-\infty}c_{\alpha}\left(n\right)q^{n}.$$

It follows from [1, Theorem 1.1] (and the obvious fact that $D^{k-1}E_{g}=g$) that

$$F_{\alpha}=\sum_{n\gg-\infty}\left(n^{k-1}c_{f}\left(n\right)-\alpha b\left(n\right)\right)$$

is a weakly holomorphic modular form of weight k. We denote by $U=U_{p}$ Atkin's U-operator. The action of this operator on formal q-series is given by

$$\left(\sum a\left(n\right)q^{n}\right)\;|U=\sum a\left(pn\right)q^{n}.$$

We are going to consider repeated application of the U-operator to the series F_{α} in the spirit of [1, Theorem 1.5].

Example (3). In the case $g = \Delta = \sum_{n\geq 1} \tau(n) q^n$ of weight 12 we choose $\alpha = c_f(1)$ as previously, and let $p = 3$ to numerically check our congruences for the first coefficients:

$$-\tfrac{359251200}{80840909811200} D^{11}\left(F_\Delta^+ - c(1)\sum_{n\geq 1}\tfrac{\tau(n)}{n^{11}}\right)\,|U_3 + \mathcal{O}(q^5) \equiv$$
$$q + 3q^2 + 9q^3 + 13q^4 + \mathcal{O}(q^5) \equiv$$
$$q - 24q^2 + 252q^3 - 1472q^4 + \mathcal{O}(q^5) = \quad \Delta \mod 3^3$$

Example (4). In the case where g is the same series defined in Example 2, choose $\alpha = c_f(1)$ as before and set $p = 3$. Then we obtain:

$$-\tfrac{3240}{2480} D^5\left(F_\Delta^+ - c(1)\sum_{n\geq 1}\tfrac{\tau(n)}{n^5}\right)\,|U_3 + O(q^{13}) \equiv$$
$$q + 69q^3 + 54q^5 + 74q^7 + 63q^9 + 54q^{11} + O(q^{13}) \equiv$$
$$q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + 540q^{11} + O(q^{13}) \mod 81 = \quad g \mod 3^4$$

Example (5). Using the same set up from Example 4, but setting $p = 5$ we see that:

$$\tfrac{3000}{4608} D^5\left(F_\Delta^+ - c(1)\sum_{n\geq 1}\tfrac{\tau(n)}{n^5}\right)\,|U_5 + O(q^{13}) \equiv$$
$$q + 113q^3 + 54q^5 + 37q^7 + 26q^9 + 40q^{11} + O(q^{13}) \equiv$$
$$q - 12q^3 + 54q^5 - 88q^7 - 99q^9 + 540q^{11} + O(q^{13}) \mod 125 = \quad g \mod 5^3$$

The examples above suggest the following theorem.

Theorem 3. *Let N be a power of a prime p, and let the character χ be trivial. Suppose that $g = \sum a(n) q^n$ is a normalized cusp Hecke eigenform, and F_g is good for g. Then there exists α such that*

$$\lim_{n\to\infty} \frac{1}{c_\alpha(p^n)} U_p^n\left(D^{k-1}(\mathcal{F}_\alpha)\right) = \breve{g},$$

where

$$\breve{g}(z) = \begin{cases} g(z) - \beta^{-1} p^{k-1} g(pz) & \text{if } N = 1 \text{ and } g \text{ is } p\text{-ordinary} \\ g(z) & \text{if } N > 1 \text{ and } g \text{ is } p\text{-ordinary} \end{cases}$$

and β is the p-adic unit root of the equation

$$X^2 - b(p) X + p^{k-1} = 0.$$

We are now in process of proving this theorem. This prove involves ideas, methods, and results from p-adic theory of modular forms.

6. Discussion of Precision of Calculations.

The code used to do the calculations was written to work in the PARI/GP computer algebra system. A C-based system can handle calculations faster than some of the more mainstream computer algebra systems on the market. There were however, a few obstacles and limitations of using this system that were encountered when calculating the various coefficients. We will now discuss the effect these have on the overall results.

GP has the capability of handling the infinite sums seen in the formulae to compute coefficients from the q-expansions. The method of calculating infinite sums however sometimes resulted in incorrect values. Upon further investigation, it was determined that when calculating an infinite sum, GP will continue to sum up terms until it encounters a preset number of consecutive zero terms at which point it will assume that any contribution from later terms will be insignificant. In some cases though, this might result in a loss of precision for certain coefficients due to the chaotic nature of the series. Another problem encountered happened when trying to calculate $c_f(5)$ of Δ. In this case, the infinite sum ran for an extended period of time because the terms did not converge to zero. Both of these problems occurred in other modular forms as well. Although a quick fix was employed by forcing GP to sum a predetermined number of terms helped correct the errors, The issue of precision now needed to be addressed.

Obviously, the larger the number of terms included in the sum, the greater the precision of the calulated value. As noted earlier, not including enough terms in a sum might result in significant loss of precision. The side effect of increasing the number of terms used also increased the amount of computing time needed to complete the calculations. A number of crude simulations were conducted to investigate what the minimum number of terms needed for precise answers was. For the first few coefficients in the q-expansion, 3000-5000 terms seemed sufficient for the purposes of this paper. It should be noted however that you need to increase the number of terms by a very large value in order to make a small increase in precision. Serveral simulations were run using an additional 30,000 terms and yielded only 4 additional digits of precision. At this many terms, the time needed to calculate one coefficient ranged from 3-5 days. In summary, we did not observe any obvious correlation between the number of terms used in the sum and the precision of the calculated coefficients. Further study of this issue is encouraged.

Even with these issues of precision, the evidence for the argument that the coefficients in question are irrational remains valid. By Theorem 2, we know that the adjusted coefficients are rational. We can then use this fact to approximate the error in the coefficients in question. In a previous finding of Ken Ono, he describes the phenomena of a recurring denominators in the continued fraction expansion of the first few coefficients which he called detectors. The presence of these detectors was believed to be evidence that the coefficients could possibly be rational. By calculating these coefficients with the code from this project, we increased the precision of the coefficients when compared to those used in Ken Ono's findings. The result is that our evidence of irrationality is stronger than his prior evidence of rationality. We further our argument by presenting a possible explanation for the detectors previously mentioned. It is highly probable that detectors arise from the fact that α is nearly rational itself. With the repeated application of the differential operator, D, it becomes plausible that detectors are a side effect of the choice of α. With lower precision, this might lead to false detectors. The details of this issue are still being studied and hopefully a complete explanation can be agreed upon.

Appendix.

The following is the code used to calculate the coefficients in this paper. It was written using the PARI/GP computer algebra system.

```
modinverse(n,p)=
{
  for(x=1,p-1,if(Mod(x*n,p)==1,return(x)));
  return(0);
```

```
}

e(a)=
{
  return(exp(2*Pi*I*a));
}

intKsum(m,n,c,k)=
{
  x=0;
  for(v=0,c-1,if(gcd(v,c)==1, x+=e((m*modinverse(v,c)+n*v)/c)));
  return(x);
}

Ksum(m,n,c,k)=
{
  x=0;
  for(v=0,c-1,if(gcd(v,c)==1,
  x+=(kronecker(c,v))^(2*k)*epsilon(v)^(2*k)*e((m*modinverse(v,c)+n*v)/c)));
  return(x);
}

Kloosterman(m,n,c,k)=
{
  if(type(k)=="t_INT", return(intKsum(m,n,c,k)), return(Ksum(m,n,c,k)));
}

c0(m,N,k)=
{
  return(-(2^k)*(Pi^k)*(I^k)*(m^(k-1))
    *suminf(c=1,Kloosterman(-m,0,c*N,2-k)/(c*N)^k));
}

c1(m,n,N,k)=
{
  return(-2*Pi*(I^k)*gamma(k)*(n/m)^((1-k)/2)
    *suminf(c=1,Kloosterman(-m,n,c*N,2-k)/(c*N)
    *besseli(k-1,(4*Pi*sqrt(abs(m*n))/(c*N)))));
}

cn(m,n,N,k)=
{
  if(n==0, return(real(c0(m,N,k))), return(real(c1(m,n,N,k))));
}
```

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