Fractal Strings, Complex Dimensions and the Spectral Operator: From the Riemann Hypothesis to Phase Transitions and Universality

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New results: Joint work with Hafedh Herichi.

Background material: In collaboration with Machiel van Frankenhuijsen (Helmut Maier and/or Carl Pomerance).

Hawaii Conference in Number Theory

Honolulu, March 7, 2012.

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$$|\eta|(A) = \sup \left\{ \sum_{k=1}^{\infty} |\eta(A_k)| \right\},$$

and $\{A_k\}_{k=1}^{\infty}$ ranges over all finite or countable partitions of A into measurable subsets of $(0, +\infty)$.

Generalized Fractal Strings

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and $\{A_k\}_{k=1}^{\infty}$ ranges over all finite or countable partitions of A into measurable subsets of $(0, +\infty)$.

By "local measure" here, we mean that η is a set function on the Borel class of $(0, +\infty)$ (with values in $[0, +\infty]$ or $\mathbb{C}$, respectively) whose restriction to any bounded Borel subset of $(0, +\infty)$ is a positive or complex measure, respectively.

Example

Consider the string consisting of the sequence of lengths $\mathcal{L} = \{a^{-j}\}_{j=1}^{\infty}$, with typically nonintegers multiplicities $w_j = b^j$, where $1 < b < a$.

Then, the measure associated to this string, called the *generalized Cantor string* (*see [La-vF3, §10.1]*) is given by

$$\eta_{\mathcal{CS}} = \sum_{j=0}^{\infty} b^j \delta_{\{a^{-j}\}}.$$

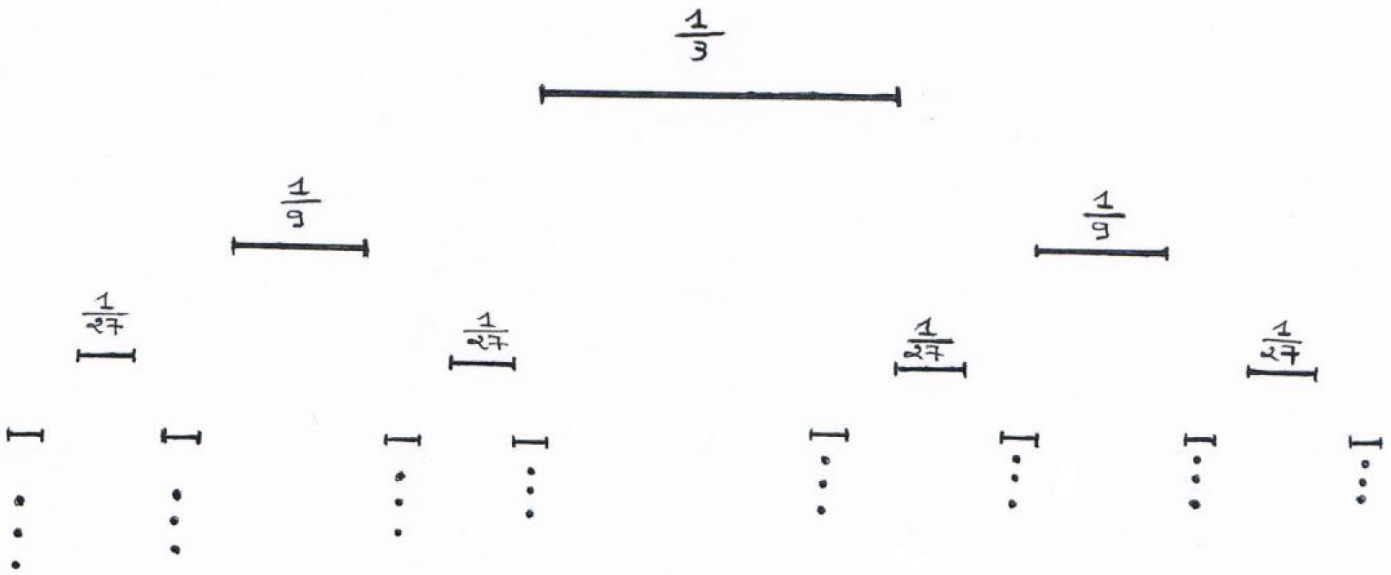
Here, δ_x is the Dirac measure concentrated at x.

Example

Let Ω be a bounded open subset of $\mathbb{R}$. Write $\Omega = \cup_{j=1}^{\infty} I_j$ as a disjoint union of (bounded, open) intervals I_j of lengths ℓ_j (repeated according to their multiplicity), and such that $\ell_j \searrow 0$ as $j \to 0$. Then Ω (or $\mathcal{L} := \{\ell_j\}_{j=1}^{\infty}$) is called an *ordinary fractal string*. Furthermore,

$$\eta_{\mathcal{L}} := \sum_{j=1}^{\infty} \delta_{\ell_j^{-1}}$$

is the associated generalized fractal string.



- The Cantor string -

Definition

Let η be a generalized fractal string. Then its *dimension*, denoted by D_η, is the abscissa of convergence of the Dirichlet integral $\int_0^{+\infty} x^{-s}\eta(dx)$; that is,

$$D_\eta = \inf\left\{\sigma \in \mathbb{R} : \int_0^{+\infty} x^{-\sigma}|\eta|(dx) < \infty\right\}.$$

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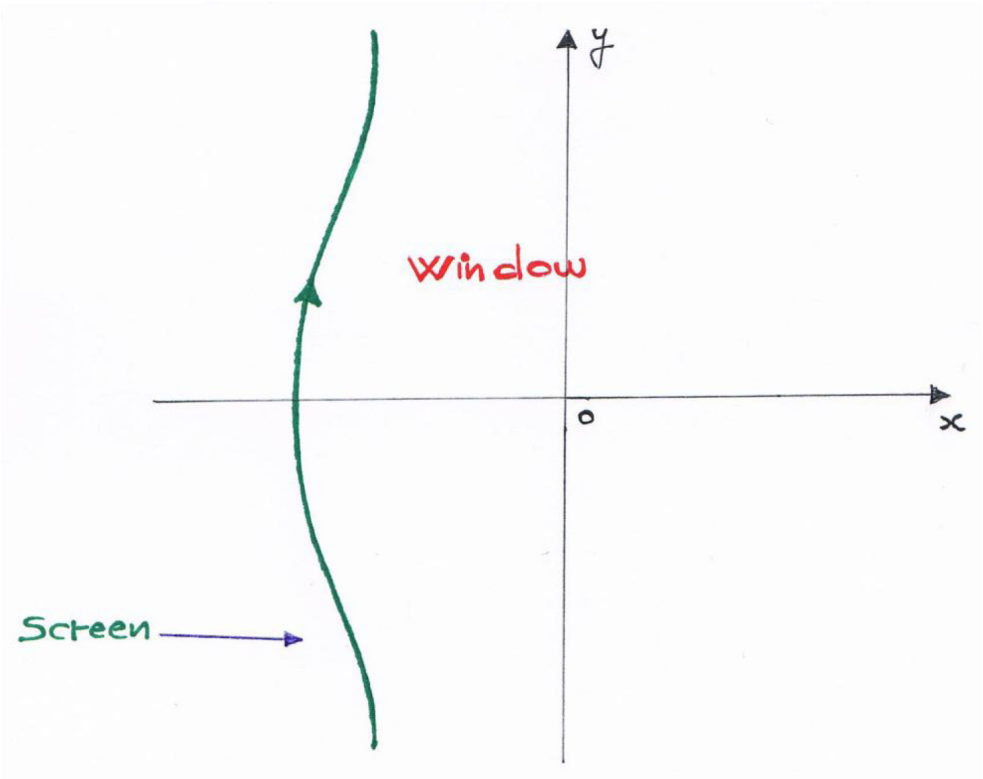
Definition

Given a generalized fractal string η, its *geometric zeta function*, denoted by ζ_η, is its Mellin transform; namely,

$$\zeta_\eta(s) = \int_0^{+\infty} x^{-s}\eta(dx) \quad \text{for } Re(s) > D_\eta\,.$$

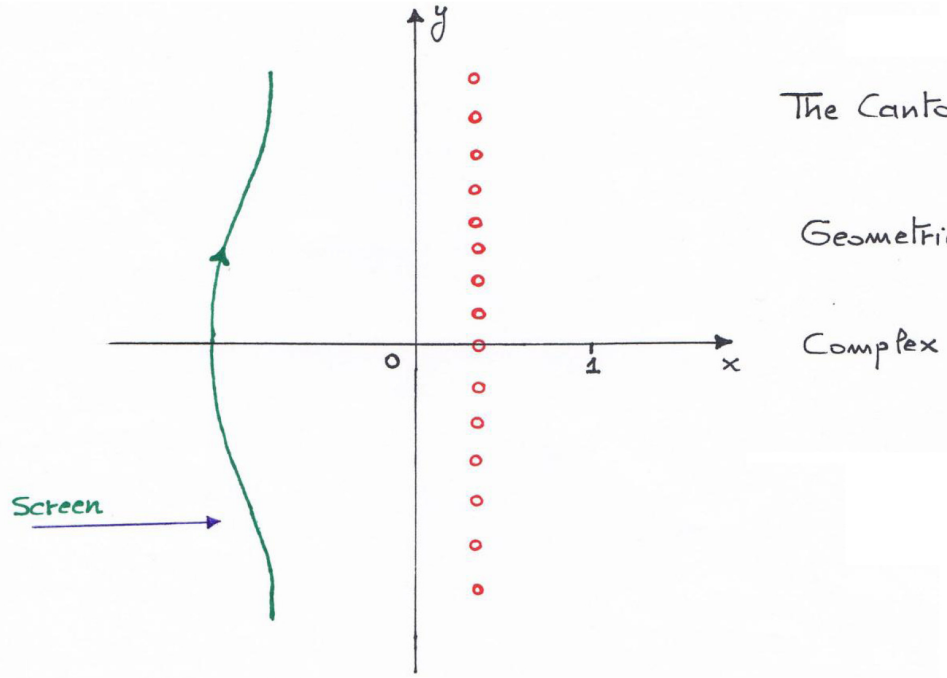

y
window
o
x
Screen

Definition

We will be interested in the meromorphic continuation of $s \mapsto \zeta_\eta(s)$. For this purpose, we define the *screen* as the curve $\mathcal{S} : S(t) + it$, $t \in \mathbb{R}$, and the *window* $\mathcal{W}$ as the subset of the complex plane $\mathcal{W} := \{ s \in \mathbb{C} : \ Re(s) \geq S(Ims)\}$.

We assume that $s \mapsto \zeta_\eta(s)$ has a meromorphic continuation to some neighborhood of $\mathcal{W}$, and we define the set of *visible complex dimensions* of η as

$$D_\eta(\mathcal{W}) = \{ \omega \in \mathcal{W} : \zeta_\eta \text{ has a pole at } \omega \}.$$



The Cantor string : $CS = \{ f_j \}_{j=1}^{\infty}$, $f_j = 3^{-j-1}$

$\omega_{f_j} = 2^j$

Geometric zeta function: $\zeta(s) = \sum_{j=1}^{+\infty} \omega_{f_j} \cdot f_j^{s} = \frac{1}{1-2\cdot 3^{s}}$

Complex dimensions : $\log_3 2 + i n \frac{2\pi}{\log 3}$

$n \in \mathbb{Z}$

Example

Harmonic string:

$$\mathfrak{h} = \sum_{j=1}^{\infty} \delta_{\{j^{-1}\}}$$

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where $p \in \mathcal{P} :=$ the set of all primes.

Note that $\mathfrak{h}$ and $\mathfrak{h}_p$ are both generalized fractal strings. Moreover, for any $p \in \mathcal{P}$, we have

$$\mathfrak{h} = \underset{p \in \mathcal{P}}{*} \mathfrak{h}_p,$$

where $*$ is the multiplicative convolution of measures on $(0, +\infty)$.

Let η and η' be two generalized fractal strings. Then we have

$$\zeta_{\eta * \eta'}(s) = \zeta_{\eta}(s).\zeta_{\eta'}(s).$$

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Hence, we have for $Re(s) > 1$,

$$\zeta_{\mathfrak{h}}(s) = \zeta_{\underset{p \in \mathcal{P}}{*} \mathfrak{h}_{\mathfrak{p}}}(s) = \zeta(s) = \prod_{p \in \mathcal{P}} \frac{1}{1 - p^{-s}} = \prod_{p \in \mathcal{P}} \zeta_{\mathfrak{h}_{\mathfrak{p}}}(s).$$

We thus recover the well-known *Euler product* for ζ.

Definition

Given a generalized fractal string η, *the spectral measure* ν *associated* to η is defined by

$$\nu(A) = \sum_{k=1}^{+\infty} \eta\left(\frac{A}{k}\right),$$

for each bounded Borel subset $A \subset (0, +\infty)$.

We define *the spectral zeta function* of η to be the geometric zeta function associated to ν; we denote it by ζ_ν.

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Lemma

Let η be a generalized fractal string. Then the spectral measure associated to η is the convolution of $\mathfrak{h}$ (the harmonic string) with η. That is,

$$\nu = \eta * \mathfrak{h}.$$

Theorem

The spectral zeta function of η, denoted by ζ_ν, is obtained by multiplying ζ_η by the Riemann zeta function

$$\zeta_\nu(s) = \zeta_\eta(s).\zeta(s).$$

Characteristics	Standard Fractal String $\mathcal{L}$	Generalized Fractal String η
Definition	A bounded open subset of $\mathbb{R}$ $\mathcal{L} = l_1, l_2, l_3, \ldots$	A local (complex or positive) measure η on $(0, +\infty)$ s.t. $\lvert\eta\rvert(0, x_0) = 0$, $x_0 > 0$
Counting Function	$N_{\mathcal{L}} = \#\{j \geq 1,\ l_j^{-1} \leq x\} = \sum_{j \geq 1,\ l_j^{-1} \leq x} 1$	$N_\eta(x) = \int_0^x d\eta = \eta(0,x) + \frac{1}{2}\eta\{x\}$
Dimension	$D_{\mathcal{L}} = \inf\{\alpha \geq 0,\ V(\varepsilon) = O(\varepsilon^{1-\alpha}) \ \varepsilon \to 0^+\}$	$D_\eta = \inf\{\sigma \in \mathbb{R},\ \zeta_{\lvert\eta\rvert}(\sigma) < +\infty\}$

Characteristics	Standard Fractal String $\mathcal{L}$	Generalized Fractal String η
Geometric Zeta Function	$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{+\infty} \ell_j^{s}$	$\zeta_{\eta}(s) = \int_0^{+\infty} x^{-s}\, \eta(dx)$ $Re(s) > D_{\eta}$
Spectral (Counting function, Measure)	$N_{\mu}(x) = \sum_{k=1}^{+\infty} N_{\mathcal{L}}\left(\frac{x}{k}\right)$	$\mu(A) = \sum_{k=1}^{+\infty} \eta\left(\frac{A}{k}\right)$
Spectral Zeta Function	$\zeta_{\mu}(s) = \sum_{f} w_f^{(\mu)} \cdot f^{-s}$ $= \zeta_{\mathcal{L}}(s) \cdot \zeta(s)$	$\zeta_{\mu}(s) = \int_0^{+\infty} x^{-s}\, \mu(dx)$ $= \zeta_{\eta}(s) \cdot \zeta_{h}(s)$

The Distributional Explicit Formulas

The Distributional Explicit Formulas

Theorem

(See [La-vF3, Ch.5]). Let η be a languid generalized fractal string. Then, for any $k \in \mathbb{Z}$, its kth distributional primitive (or anti-derivative) $\mathcal{P}_\eta^{[k]}$ is given by

$$\mathcal{P}_\eta^{[k]}(x) = \sum_{\omega \in D_\eta(\mathcal{W})} res\left(\frac{x^{s+k-1}\zeta_\eta(s)}{(s)_k}; \omega\right) + \frac{1}{(k-1)!} \cdot \sum_{j=0}^{k-1} C_j^{k-1}(-1)^j x^{k-1-j}\zeta_\eta(-j) + \mathcal{R}_\eta^{[k]}(x),$$

where $\mathcal{R}_\eta^{[k]}(x) = \frac{1}{2\pi i}\int_{\mathcal{S}} x^{s+k-1}\zeta_\eta(s)\frac{ds}{(s)_k}$ is the error term and can be suitably estimated as $x \mapsto +\infty$.

In addition, if η is strongly languid, then we may choose $\mathcal{W} = \mathbb{C}$ and $\mathcal{R}_\eta(x) \equiv 0$.

When we apply the distributional explicit formulas at level $k = 0$, assuming that η is a languid generalized fractal string whose complex dimensions are simple and satisfies certain mild additional conditions, we obtain that, as a distribution, the measure η is given by the following *density of geometric states* (or *density of lengths*) formula:

$$\eta = \sum_{\omega \in \mathcal{D}_\eta(\mathcal{W})} res(\zeta_\eta(s); \omega) x^{\omega - 1}.$$

We also obtain for the spectral measure, by applying the previous theorem to $\nu = \mathcal{P}_\nu^{[0]}$, the following *density of spectral states* (or *density of frequencies*) formula:

$$\begin{aligned}\nu &= \zeta_\eta(1) + \sum_{\omega \in \mathcal{D}_\eta(\mathcal{W})} res(\zeta_\eta(s)\zeta(s)x^{s-1}; \omega)x^{\omega-1} \\ &= \zeta_\eta(1) + \sum_{\omega \in \mathcal{D}_\eta(\mathcal{W})} \zeta(\omega) res(\zeta_\eta(s); \omega)x^{\omega-1},\end{aligned}$$

for simple poles.

The Inverse Spectral Problem

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The inverse problem for fractal strings considered in [LaMa1, LaMa2] (and later revisited in [La-vF2, La-vF3]) is the following, for any fixed $D \in (0, 1)$:

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$(IVS)_D$ *Given any fractal string $\mathcal{L}$ of dimension D such that for some constants C_D and $\delta > 0$,*

$$N_\nu(x) = W(x) - c_D x^D + O(x^{D-\delta}),$$

as $x \to +\infty$, is it true that $\mathcal{L}$ is Minkowski measurable?

(Here, the leading term $W(x)$ is the *Weyl term*, given by $W(x) := vol_1(\mathcal{L})x$, and below, $\mathcal{M}$ is the *Minkowski content* of $\mathcal{L}$.)

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Remark:

It follows from results in [LaPo2] and [La2, La3] that if such a nonzero constant c_D exists, then $c_D > 0$ and is given by

$$c_D = 2^{-(1-D)}(1-D)(-\zeta(D))\mathcal{M}.$$

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For any given $D \in (0,1)$, the inverse spectral problem $(IVS)_D$ has an affirmative answer if and only if $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ such that $Re(s) = D$. Hence, $(IVS)_D$ is not true in the "mid-fractal" case when $D = \frac{1}{2}$, and it holds everywhere else (i.e, for every $D \in (0,1)$, $D \neq \frac{1}{2}$) if and only if the Riemann hypothesis is true.

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This spectral reformulation was revisited in [La-vF2, La-vF3] by using the then rigorously developed theory of complex dimensions and the associated explicit formulas. (See [La-vF3,Ch.9].)

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and at level $k = 1$ as the map

$$N_\eta(x) \longmapsto \nu(N_\eta)(x) = N_\nu(x) = \sum_{k=1}^{\infty} N_\eta\left(\frac{x}{k}\right).$$

For every prime number p, we also define the *p-factor* of ν by

$$N_\eta(x) \longmapsto \nu_p(N_\eta)(x) = N_{\nu_p}(x) = N_{\eta * h_p}(x) = \sum_{k=0}^{\infty} N_\eta(xp^{-k}),$$

where the terms in the sum necessarily vanish when $p^k \geq x$.

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where the terms in the sum necessarily vanish when $p^k \geq x$.

The operators ν_p commute with each other and their composition gives the Euler product for ν :

$$N_\eta(x) \longmapsto \nu(N_\eta)(x) = \Big(\prod_{p \in \mathcal{P}} \nu_p\Big)(N_\eta).$$

Making the change of variable $x = e^t$ $(x > 0)$ or equivalently, $t = \log x$ (and hence, $t \in \mathbb{R}$), and writing $f(t) = N_\eta(x)$, we obtain the *additive version* of the *spectral operator*

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$$f(t) \mapsto \mathfrak{a}(f)(t) = \sum_{k=1}^{\infty} f(t - \log k),$$

and of its operator-valued *Euler factors* (for each prime $p \in \mathcal{P}$)

$$f(t) \mapsto \mathfrak{a}_p(f)(t) = \sum_{k=0}^{\infty} f(t - k \log p).$$

The spectral operator $\mathfrak{a}$ and its Euler factors $\mathfrak{a}_p$ are also related by an operator-valued *Euler product*

$$f(t) \longmapsto \mathfrak{a}(f)(t) = \Big(\prod_{p \in \mathcal{P}} \mathfrak{a}_p \Big)(f)(t),$$

where the product is given in the sense of the composition of operators.

If we denote by $\partial := \frac{d}{dt}$ the first order differential operator with respect to t, the Taylor series associated to f, a smooth function, can be written as

$$\begin{aligned} f(t+h) &= f(t) + \frac{hf'(t)}{1!} + \frac{h^2 f''(t)}{2!} + \ldots \\ &= e^{h\frac{d}{dt}}(f)(t) = e^{h\partial}(f)(t); \end{aligned}$$

that is, $\partial = \frac{d}{dt}$ is the infinitesimal generator of the (one-parameter) group of shifts on the real line.

This gives a new representation for the spectral operator and its prime factors:

$$\begin{aligned}
\mathfrak{a}(f)(t) &= \sum_{k=1}^{\infty} e^{-(\log k)\partial}(f)(t) \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k^{\partial}}\right)(f)(t) \\
&= \zeta(\partial)(f)(t) \\
&= \prod_{p\in\mathcal{P}} (1-p^{-\partial})^{-1}(f)(t),
\end{aligned}$$

and for any prime p,

$$
\begin{aligned}
\mathfrak{a}_p(f)(t) &= \sum_{k=0}^{\infty} f(t - k\log p) \\
&= \sum_{k=0}^{\infty} e^{-k(\log p)\partial}(f)(t) \\
&= \sum_{k=0}^{\infty} \left(p^{-k\partial}\right)(f)(t) \\
&= (1 - p^{-\partial})^{-1}(f)(t) \\
&= \zeta_{h_p}(\partial)(t).
\end{aligned}
$$

The Weighted Hilbert Space $\mathbb{H}_c$

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For $c \geq 0$, let

$$\mathcal{H}_c := \left\{ f \in C^\infty(\mathbb{R}) : supp(f) \subset (0, +\infty) \text{ and } \int_0^{+\infty} |f(t)|^2 e^{-2ct} dt < \infty \right\}$$

$\mathcal{H}_c$ is a pre-Hilbert space for the natural inner product indicated below.

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$\mathcal{H}_c$ is a pre-Hilbert space for the natural inner product indicated below.

Its completion is a Hilbert space and is denoted by $\mathbb{H}_c$. It is equipped with the following inner product

$$< f, g >_c = \int_0^\infty f(t)\overline{g(t)} e^{-2ct} dt$$

and the associated Hilbert norm $||.||_c = \sqrt{< ., . >_c}$ (so that $||f||_c^2 = \int_0^{+\infty} |f(t)|^2 e^{-2ct} dt$).

The Differentiation Operator $A = \partial_c$

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Given $c \geq 0$, we define $A := \partial = \partial_c = \frac{d}{dt}$ as the unbounded linear operator from $\mathbb{H}_c$ to itself with domain $D(A)$ consisting of all the functions $f \in \mathbb{H}_c$ that are (locally) absolutely continuous on $\mathbb{R}$ *(i.e., $f \in C_{abs}(\mathbb{R})$)* and such that $f' \in \mathbb{H}_c$ (where f' denotes the pointwise derivative of f, which exists Lebesgue almost everywhere on $\mathbb{R}$). Furthermore, for $f \in D(A)$, we let

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$$Af = \partial f := f', \quad \text{for all } f \in D(A).$$

It follows from the definition of $\mathbb{H}_c$, $D(A)$ and a well-known lemma (about absolutely continuous functions) that every $f \in D(A)$ naturally satisfies the following *boundary conditions* at $-\infty$ and $+\infty$, respectively:

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(2) (Boundary condition at $+\infty$) $\lim_{t \to +\infty} f(t)e^{-tc} = 0$.

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Remark:

Intuitively, condition (2) means that the corresponding fractal strings have (Minkowski) dimension $D \leq c$. (See [LapPo2].)

Normality of the unbounded operator ∂_c

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Theorem

For every $c \geq 0$, $A = \partial_c$ is an unbounded normal linear operator on $\mathbb{H}_c$.

Moreover, its adjoint A^* is given by $A^* = 2c - A$, with $D(A^*) = D(A)$.

Spectrum of ∂_c

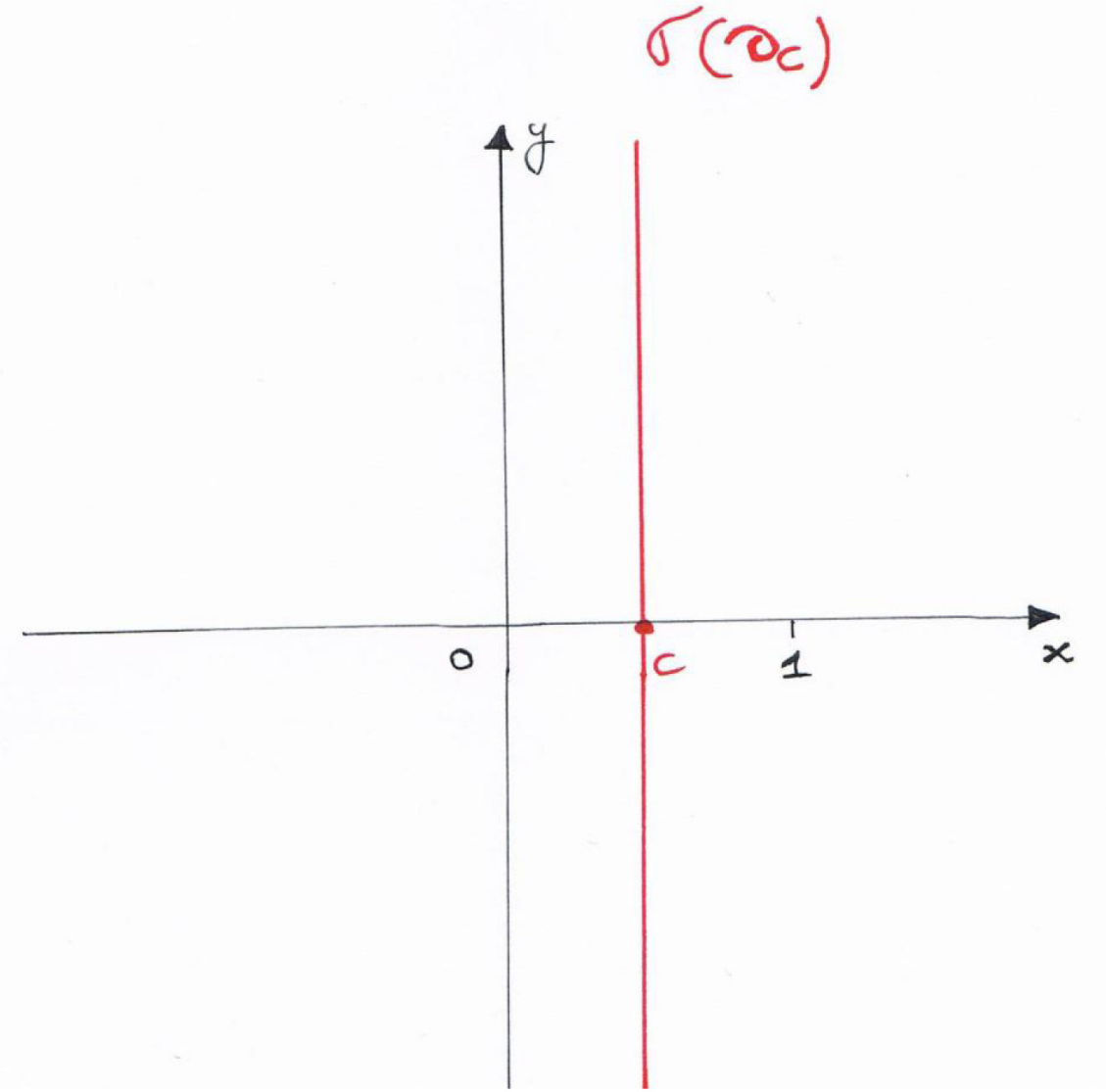
Spectrum of ∂_c

Theorem

For every $c \geq 0$, the spectrum $\sigma(A)$ of the differentiation operator $A = \partial_c$ is the closed vertical line of the complex plane passing through c. Furthermore, it is equal to the essential spectrum $\sigma_e(A)$ of A:

$$\sigma(A) = \sigma_e(A) = \{\, \lambda \in \mathbb{C} |\, Re(\lambda) = c \,\}.$$

Moreover, the point spectrum of A is empty (i.e., A does not have any eigenvalues).


$\sigma(\mathcal{D}_c)$
y
x
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c
1

The strongly continuous group $e^{-t\partial_c}$

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Lemma

For every $c \geq 0$, $\{e^{-t\partial_c}\}_{t\in\mathbb{R}}$ is a strongly continuous group of (bounded linear) operators and

$$||e^{-t\partial_c}|| = e^{-ct}$$

for any $t \in \mathbb{R}$.

Moreover, its adjoint group $\{(e^{-t\partial_c})^*\}_{t\in\mathbb{R}}$ is given by

$$\{e^{-t\partial_c^*}\}_{t\in\mathbb{R}} = \{e^{-t(2c-\partial_c)}\}_{t\in\mathbb{R}} = e^{-2ct}\{e^{t\partial_c}\}_{t\in\mathbb{R}}.$$

Lemma

The strongly continuous group of operators $\{e^{-t\partial}\}_{t\in\mathbb{R}}$ *is a translation (or shift) group. That is, for every* $t \in \mathbb{R}$,

$$(e^{-t\partial})(f)(u) = f(u-t)$$

for all $f \in \mathbb{H}_c$ *and* $u \in \mathbb{R}$. *(For a fixed* $t \in \mathbb{R}$, *this equality holds for elements in* $\mathbb{H}_c$ *and hence, for a.e.* $u \in \mathbb{R}$.*)*

Lemma

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$$(e^{-t\partial})(f)(u) = f(u-t)$$

for all $f \in \mathbb{H}_c$ *and* $u \in \mathbb{R}$. *(For a fixed* $t \in \mathbb{R}$, *this equality holds for elements in* $\mathbb{H}_c$ *and hence, for a.e.* $u \in \mathbb{R}$.*)*

Remark: As a result, $\partial = \partial_c$, the infinitesimal generator of the shift group $\{e^{-t\partial}\}_{t\in\mathbb{R}}$, is called the *infinitesimal shift* of the real line (with parameter $c \geq 0$).

The Spectrum of $\mathfrak{a}$

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We can now define the *spectral operator* as follows:

$$\mathfrak{a} = \zeta(\partial),$$

via the measurable functional calculus for unbounded normal operators. (If, for simplicity, we assume $c \neq 1$ in order to avoid the pole of ζ at $s = 1$, then ζ is holomorphic in a neighborhood of $\sigma(\partial)$. If $c = 1$ is allowed, then we may also use the meromorphic functional calculus for sectorial operators; see [Haase].)

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Theorem

Assume that $c > 1$. Then, for any $f \in D(\mathfrak{a})$, we have

$$\mathfrak{a}(f)(t) = \sum_{k=1}^{\infty} f(t - \log k) = \zeta(\partial)(f)(t) = \sum_{n=1}^{\infty} n^{-\partial}(f)(t).$$

Remark:

For any $c > 0$, we also show that the above equation holds for all f in a suitable dense subspace of $D(\mathfrak{a})$.

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Theorem

Assume that $c \geq 0$. Then

$$\sigma(\mathfrak{a}) = \overline{\zeta(\sigma(\partial))} = cl\left(\zeta(\{\lambda \in \mathbb{C} |\, Re(\lambda) = c\})\right),$$

where $\sigma(\mathfrak{a})$ is the spectrum of $\mathfrak{a}$ and $\overline{N} = cl(N)$ is the closure of $N \subset \mathbb{C}$.

Quasi-Invertibility of the Spectral Operator and a Spectral Reformulation of RH

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Theorem

Assume that $c \geq 0$. Then, the spectral operator $\mathfrak{a} = \zeta(\partial)$ is quasi-invertible if and only if the Riemann zeta function does not vanish on the vertical line $\{s \in \mathbb{C} : Re(s) = c\}$.

Corollary

The spectral operator $\mathfrak{a}$ *is quasi-invertible for all* $c \in (0,1) - \frac{1}{2}$ *if and only if the Riemann hypothesis is true.*

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Remarks:

- The notion of quasi-invertibility will be defined in the next part.

Corollary

The spectral operator $\mathfrak{a}$ is quasi-invertible for all $c \in (0,1) - \frac{1}{2}$ if and only if the Riemann hypothesis is true.

Remarks:

- The notion of quasi-invertibility will be defined in the next part.
- It suffices to require $c \in (0, \frac{1}{2})$ (or $c \in (\frac{1}{2}, 1)$) in the above corollary. This follows from the functional equation for ζ connecting $\zeta(s)$ and $\zeta(1-s)$.

Truncated Spectral Operators and Quasi-Invertibility

We show in [HerLa1] that for any $c \geq 0$, the infinitesimal shift $\partial = \partial_c$ is given by

$$\partial = c + iV,$$

where V is an unbounded self-adjoint operator such that $\sigma(V) = \mathbb{R}$. Thus, given $T \geq 0$, we define the *truncated infinitesimal shift* as follows:

$$A^{(T)} = \partial^{(T)} := c + iV^{(T)},$$

where

$$V^{(T)} := \phi^{(T)}(V)$$

(in the sense of the functional calculus),

and $\phi^{(T)}$ is a suitable (i.e., T-admissible) continuous (if $c \neq 1$) or meromorphic (if $c = 1$) cut-off function (chosen so that $\overline{\phi^{(T)}(\mathbb{R})} = \sigma\left(A^{(T)}\right) = c + i[-T, T]$).

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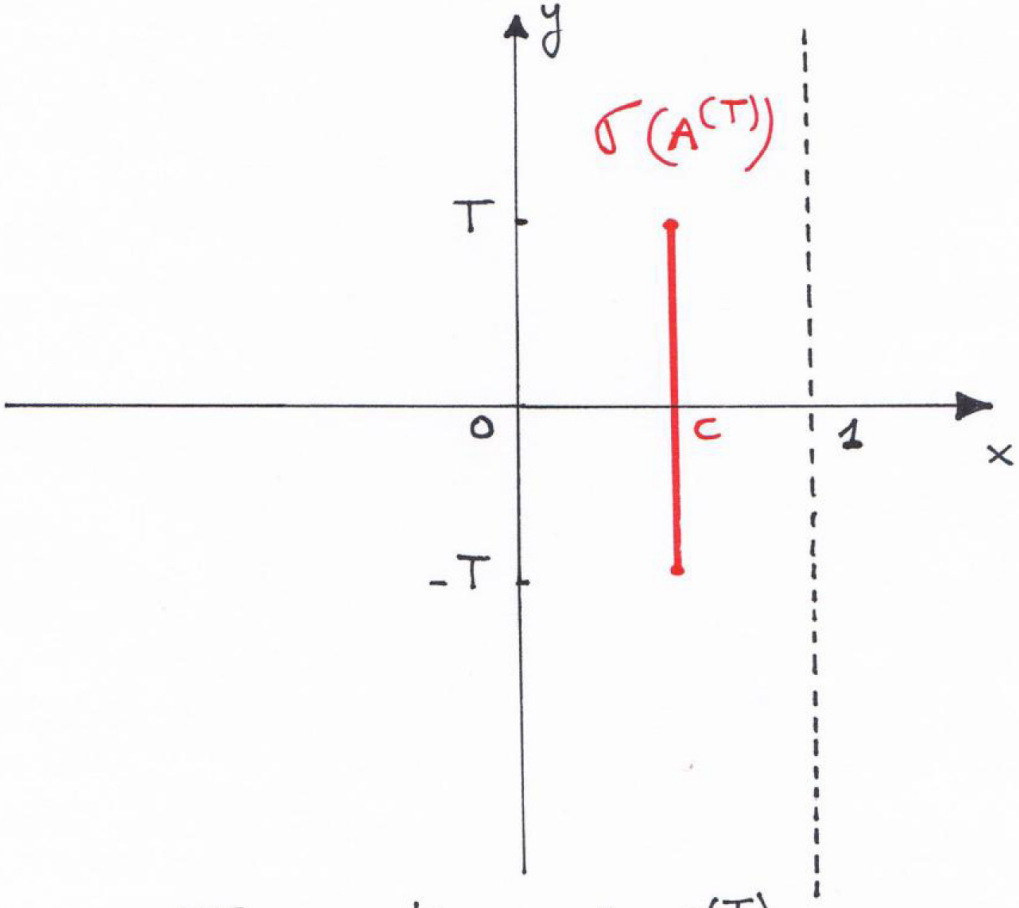
Similarly, the *truncated spectral operator* is defined (also for $c \geq 0$) by

$$\mathfrak{a}^{(T)} := \zeta\left(\partial^{(T)}\right).$$



- The spectrum of $A^{(T)}$ -

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(i) If $c \neq 1$, $\phi^{(T)}$ is any continuous function such that $\overline{\phi^{(T)}}(\mathbb{R}) = [-T, T]$. (For example, $\phi^{(T)}(\tau) = \tau$ for $0 \leq \tau \leq T$ and $= T$ for $\tau \geq T$; also, $\phi^{(T)}$ is odd.)

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(ii) If $c = 1$ (which corresponds to the pole of $\zeta(s)$ at $s = 1$), then $\phi^{(T)}$ is a suitable meromorphic analog of (i). (For example, $\phi^{(T)}(s) = \frac{2T}{\pi} \tan^{-1}(s)$, so that $\overline{\phi^{(T)}(\mathbb{R})} = [-T, T]$.)

One then uses the measurable functional calculus and an appropriate (continuous or meromorphic, for $c \neq 1$ or $c = 1$, respectively) version of the Spectral Mapping Theorem (**SMT**) for unbounded normal operators in order to define both $\partial^{(T)}$ and $\mathfrak{a}^{(T)} = \zeta(\partial^{(T)})$, as well as to determine their spectra.

$$\textbf{SMT}: \ \sigma(\psi(L)) = \overline{\psi(\sigma(L))}$$

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Note that for $c \neq 1$ (resp., $c = 1$), $\partial^{(T)}$ and $\mathfrak{a}^{(T)}$ are then continuous (resp., meromorphic) functions of the normal (and sectorial, see [Haa]) operator ∂. An entirely analogous statement is true for the spectral operator $\mathfrak{a} = \zeta(\partial)$.

The above construction can be generalized as follows:

Given $0 \leq T_0 \leq T$, one can define a (T_0, T)-*admissible cut-off function* $\phi^{(T_0,T)}$ exactly as above, except with $[-T, T]$ replaced with $\{\tau \in \mathbb{R} : \ T_0 \leq |\tau| \leq T\}$.

Correspondingly, one can define $V^{(T_0,T)} = \phi^{(T_0,T)}(V)$,

$$A^{(T_0,T)} = \partial^{(T_0,T)} := c + iV^{(T_0,T)}$$

and

$$\mathfrak{a}^{(T_0,T)} = \zeta(\partial^{(T_0,T)}),$$

where $\partial^{(T_0,T)}$ is the (T_0, T)-*infinitesimal shift* and $\mathfrak{a}^{(T_0,T)}$ is the (T_0, T)-*truncated spectral operator*.

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Remark: Note that for $T_0 = 0$, we recover $A^{(T)}$ and $\mathfrak{a}^{(T)}$.

Definition

The spectral operator $\mathfrak{a}$ is *quasi-invertible* if its truncation $\mathfrak{a}^{(T)}$ is invertible for all $T \geq 0$.

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Remark:

In the definition of "almost invertibility", T_0 is allowed to depend on the parameter c.

Note:

$$\text{quasi-invertible} \Rightarrow \text{almost invertible.}$$

Theorem

For all $T \geq 0$, $A^{(T)}$ and $\mathfrak{a}^{(T)}$ are bounded normal operators, with spectra respectively given by

$$\sigma\left(A^{(T)}\right) = \{c + i\tau : |\tau| \leq T\}$$

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- Recall that $\mathfrak{a}^{(T)}$ is invertible if and only if $0 \notin \sigma\left(\mathfrak{a}^{(T)}\right)$.

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Remarks:

- Recall that $\mathfrak{a}^{(T)}$ is invertible if and only if $0 \notin \sigma\left(\mathfrak{a}^{(T)}\right)$.
- More generally, given $0 \leq T_0 \leq T$, the exact counterpart of the above theorem holds for $A^{(T_0,T)}$ and $\mathfrak{a}^{(T_0,T)}$, except with $|\tau| \leq T$ replaced with $T_0 \leq |\tau| \leq T$.

Corollary

Assume that $c \geq 0$. Then, the truncated spectral operator $\mathfrak{a}^{(T)}$ is invertible if and only if ζ does not vanish on the vertical line segment $\{s \in \mathbb{C} : \ Re(s) = c, \, |Im(s)| \leq T\}$.

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Remark:

Naturally, given $0 \leq T_0 \leq T$, the same result is true for $\mathfrak{a}^{(T_0,T)}$ provided $|Im(s)| \leq T$ is replaced with $T_0 \leq |Im(s)| \leq T$.

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Theorem

Assume that $c \geq 0$. Then,

1. $\mathfrak{a}$ *is quasi-invertible if and only if* ζ *does not vanish (i.e., does not have any zeroes) on the vertical line* $Re(s) = c$.
2. $\mathfrak{a}$ *is almost invertible if and only if all but (at most) finitely many zeroes of* ζ *are off the vertical line* $Re(s) = c$.

Corollary

1. $\mathfrak{a}$ *is quasi-invertible for all* $c \in (\frac{1}{2}, 1)$ *if and only if the Riemann hypothesis is true.*

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2. $\mathfrak{a}$ *is almost invertible for all* $c \in (\frac{1}{2}, 1)$ *if and only if the Riemann hypothesis (RH) is "almost true "(i.e., on every vertical line* $Re(s) = c$, $c > \frac{1}{2}$, *there are at most finitely many exceptions to RH).*

Remark:

According to our previous results, we have (for the spectral operator $\mathfrak{a}$):

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3. For $c > 1$, $\mathfrak{a}$ is quasi- (and hence, almost) invertible. In fact, we will next see that $\mathfrak{a}$ is also invertible for $c > 1$.

Spectrum of $\mathfrak{a}$ Revisited: Zeta Values and Universality

*Recall that, by the Spectral Mapping Theorem (***SMT***) for unbounded normal operators, $\sigma(\mathfrak{a})$ (the spectrum of $\mathfrak{a}$) is equal to the closure of the range of ζ on the vertical line* $\sigma(A) = \{Re(s) = c\}$. Hence, $\sigma(\mathfrak{a})$ is equal to $\{\zeta(c + i\tau) : \tau \in \mathbb{R}\}$ union its limit points (in $\mathbb{C}$).

Also, by definition of $\sigma(\mathfrak{a})$, $\mathfrak{a}$ *is invertible if and only if* $0 \notin \sigma(\mathfrak{a})$.

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2. (*Universality*) *For* $c \in (\frac{1}{2}, 1)$, $\sigma(\mathfrak{a}) = \mathbb{C}$. (*This follows from the Bohr–Landau Theorem and, more generally, from the universality of* ζ *in the right critical strip* $\frac{1}{2} < Re(s) < 1$.) *Hence,* $\mathfrak{a}$ *is not invertible (because* $0 \in \sigma(\mathfrak{a})$*).*

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3. *For* $c \in (0, \frac{1}{2})$, $\sigma(\mathfrak{a})$ *is unbounded and conditionally (i.e., under RH),* $\sigma(\mathfrak{a}) \neq \mathbb{C}$ *and, in particular,* $0 \notin \sigma(\mathfrak{a})$, *so that* $\mathfrak{a}$ *is invertible.*

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3. *For $c \in (0, \frac{1}{2})$, $\sigma(\mathfrak{a})$ is unbounded and conditionally (i.e., under RH), $\sigma(\mathfrak{a}) \neq \mathbb{C}$ and, in particular, $0 \notin \sigma(\mathfrak{a})$, so that $\mathfrak{a}$ is invertible.*

Remark:

The last statement in the third part of the theorem follows from the non-universality of ζ in the left critical strip $0 < Re(s) < \frac{1}{2}$; see [KaSt].

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1. What is $\sigma(\mathfrak{a})$ when $c \in (0, \frac{1}{2})$? (It is a very complicated, closed, unbounded, and (conditionally) strict subset of $\mathbb{C}$.) Is RH needed for $\sigma(\mathfrak{a}) \neq \mathbb{C}$ to be true? (Compare [KaSt].)

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2. Does $0 \notin \sigma(\mathfrak{a})$, when $c \in (0, \frac{1}{2})$? Unconditionally (or else under the Lindelöf hypothesis), we conjecture that $0 \notin \sigma(\mathfrak{a})$ and hence, that $\mathfrak{a}$ is invertible for $0 < c < \frac{1}{2}$.

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3. Conjecturally, for $c = \frac{1}{2}$, we have that $\sigma(\mathfrak{a}) = \mathbb{C}$. Moreover, $\mathfrak{a}$ is clearly not invertible for $c = \frac{1}{2}$ since ζ has zeroes on the critical line and hence, we know that $0 \in \sigma(\mathfrak{a})$.

Spectra, Riemann Zeroes, and Phase Transitions

I. Phase Transition at $c = \frac{1}{2}$

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Theorem

Conditionally (i.e., under RH), the spectral operator $\mathfrak{a}$ is quasi-invertible for all $c \neq \frac{1}{2}$ ($c \in (0,1)$), and (unconditionally) it is not quasi-invertible (not even almost invertible) for $c = \frac{1}{2}$.

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Remark:

1. Recall that $\mathfrak{a}$ is quasi-invertible for all $c \neq \frac{1}{2} \iff$ RH.
2. Furthermore, $\mathfrak{a}$ is not almost invertible for $c = \frac{1}{2}$ because ζ has infinitely many zeroes on the critical line $Re(s) = \frac{1}{2}$.

II. Phase Transitions at $c = \frac{1}{2}$ and $c = 1$

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Theorem

The spectral operator $\mathfrak{a}$ is invertible for $c \geq 1$, is not invertible for $\frac{1}{2} < c < 1$ (conjecturally, also for $c = \frac{1}{2}$), and invertible (conditionally) for $0 < c < \frac{1}{2}$.

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Theorem

The spectrum $\sigma(\mathfrak{a})$ is non-compact (and hence, unbounded), but (conditionally) not all of $\mathbb{C}$ for $0 < c < \frac{1}{2}$. It is all of $\mathbb{C}$ for $\frac{1}{2} < c < 1$, and compact (and thus, bounded) for $c > 1$.

III. Phase Transition at $c = 1$

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Theorem

Unconditionally, the spectral operator $\mathfrak{a}$ is unbounded for $c < 1$, and is bounded for $c > 1$.

Universality of the Riemann Zeta Function $\zeta = \zeta(s)$

The "universality" of ζ roughly means that any non-vanishing holomorphic function in $\{\frac{1}{2} < Re(s) < 1\}$ can be approximated arbitrarily closely by imaginary translates of ζ.

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More precisely, we have the following well-known remarkable result (Extended Voronin Theorem).

Theorem

Let K be any compact subset of $\{\frac{1}{2} < Re(s) < 1\}$, with connected complement in $\mathbb{C}$. Let $g : K \to \mathbb{C}$ be a non-vanishing continuous function that is holomorphic in the interior of K (which may be empty). Then, given any $\epsilon > 0$, there exists $\tau \geq 0$ (depending only on ϵ) such that

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$$J_{sc}(\tau) := \max_{s \in K} |g(s) - \zeta(s + i\tau)| \leq \epsilon.$$

In fact, the set of such τ's has a positive lower density and, in particular, is infinite. More precisely, we have

$$\liminf_{\rho \to +\infty} \frac{1}{\rho} \, vol_1 \left(\{ \tau \in [0, \rho] : \; J_{sc}(\tau) \leq \epsilon \} \right) > 0.$$

Remarks:

- Voronin's Original Universality Theorem (1975) corresponds to the choice of $K := \overline{D}(\frac{3}{4}, r)$, the closed disk of center $\frac{3}{4}$ and radius r, with $0 < r < \frac{1}{4}$ arbitrary.

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- If we assume the Riemann hypothesis, then $\zeta(s)$ does not have any zeroes in $\frac{1}{2} < Re(s) < 1$. Hence, applying the Universality Theorem to $g(s) := \zeta(s)$ and upon some elementary manipulations, one sees that *scaled copies* of ζ *can be found within itself at all scales. In other words, conditionally, the Riemann zeta function is both fractal and chaotic*.

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The "universality" of the spectral operator $\mathfrak{a} = \zeta(\partial)$ roughly means that any non-vanishing holomorphic function of ∂ in $\{\frac{1}{2} < Re(s) < 1\}$ can be approximated arbitrarily closely by imaginary translates of $\zeta(\partial)$.

Universality of the Spectral Operator $\mathfrak{a} = \zeta(\partial)$

The "universality" of the spectral operator $\mathfrak{a} = \zeta(\partial)$ roughly means that any non-vanishing holomorphic function of ∂ in $\{\frac{1}{2} < Re(s) < 1\}$ can be approximated arbitrarily closely by imaginary translates of $\zeta(\partial)$.

More precisely, we have the following *operator-theoretic generalization of the Extended Voronin Universality Theorem*, expressed in terms of the imaginary translates of the T-*truncated spectral operators* $\mathfrak{a}^{(T)} = \zeta(\partial^{(T)})$, where $\partial^{(T)} = \partial_c^{(T)}$ is the T-*truncated infinitesimal shift* (with parameter c).

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Theorem

Let K be a compact subset of $\{\frac{1}{2} < Re(s) < 1\}$ of the following form. Assume, for simplicity, that $K = \mathcal{K} \times [-T_0, T_0]$, for some $T_0 \geq 0$, where $\mathcal{K}$ is a compact subset of $(\frac{1}{2}, 1)$.

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Let $g : K \to \mathbb{C}$ be a non-vanishing continuous function that is holomorphic in the interior of K (which may be empty). Then, given any $\epsilon > 0$, there exists $\tau \geq 0$ (depending only on ϵ) such that

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$$\mathcal{H}_{op}(\tau) := \sup_{c \in \mathcal{K},\, 0 \leq T \leq T_0} \left|\left| g\left(\partial_c^{(T)}\right) - \zeta\left(\partial_c^{(T)} + i\tau\right) \right|\right| \leq \epsilon,$$

where $\partial^{(T)} = \partial_c^{(T)}$ is the T-truncated infinitesimal shift (with parameter c) and $||.||$ is the norm in $\mathcal{B}(\mathbb{H}_c))$ (the space of bounded linear operators on $\mathbb{H}_c$).

In fact, the set of all such τ's has a positive lower density and, in particular, is infinite. More precisely, we have

$$\liminf_{\rho \to +\infty} \frac{1}{\rho} \, vol_1 \left(\{ \tau \in [0, \rho] : \mathcal{H}_{op}(\tau) \leq \epsilon \} \right) > 0.$$

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Remark:

A remarkable feature of the above generalization is the uniformity in the parameter $c \in \mathcal{K}$ and in $T \in [0, T_0]$ of the stated approximation of $g\left(\partial_c^{(T)}\right)$.

We will next state a further generalization of the *operator-theoretic Extended Voronin Universality Theorem*. For pedagogical reasons, we will choose assumptions (on the compact set K) that simplify its formulation. (The appropriate definitions and possible extensions will be given just after the statement of the theorem.)

Theorem

Let K be any compact, vertically convex, subset of $\{\frac{1}{2} < Re(s) < 1\}$, with connected complement in $\mathbb{C}$. Assume, for simplicity, that K is symmetric with respect to the real axis. Denote by $\mathcal{K}$ the projection of K onto the real axis, and for $c \in \mathcal{K}$, let

$$T(c) := \sup\left(\{T \geq 0 : [c - iT, c + iT] \subset K\}\right).$$

(By construction, $\mathcal{K}$ is a compact subset of $(\frac{1}{2}, 1)$ and $0 \leq T(c) < \infty$, for $c \in \mathcal{K}$.) Assume further that $c \mapsto T(c)$ is continuous on $\mathcal{K}$.

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$$\mathcal{J}_{op}(\tau) := \sup_{c \in \mathcal{K},\, 0 \leq T \leq T(c)} \left\| g\left(\partial_c^{(T)}\right) - \zeta\left(\partial_c^{(T)} + i\tau\right) \right\| \leq \epsilon,$$

where $\partial^{(T)} = \partial_c^{(T)}$ *is the* T*-truncated infinitesimal shift (with parameter* c*) and* $||.||$ *denotes the usual norm in* $\mathcal{B}(\mathbb{H}_c)$ *(the space of bounded linear operators on* $\mathbb{H}_c$*).*

In fact, the set of such τ'*s has a positive lower density and, in particular, is infinite. More precisely*, we have

$$\liminf_{\rho \to +\infty} \frac{1}{\rho} vol_1(\{\tau \in [0, \rho] : \mathcal{J}_{op}(\tau) \leq \epsilon\}) > 0.$$

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- Instead of assuming that K is symmetric with respect to the real axis, it would suffice to suppose that $c + iT \in K$ (for some $c \in \mathcal{K}$ and $T > 0$) implies that $c - iT \in K$, and vice versa.
- As in the scalar case (and taking K to be a line segment), we see that any continuous curve $\left(\text{of } \partial_c^{(T)}\right)$ can be approximated by imaginary translates of $\mathfrak{a}^{(T)} = \zeta\left(\partial_c^{(T)}\right)$. Hence, roughly speaking, *the spectral operator* $\mathfrak{a}$ $\left(\text{or its } T\text{-truncations } \mathfrak{a}^{(T)}\right)$ *can emulate any type of complex behavior: it is chaotic*.

- Conditionally (i.e., under RH), and applying the above operator-theoretic Universality Theorem to $g(s) := \zeta(s)$, we see that, roughly speaking, arbitrarily small scaled copies of the spectral operator are encoded within $\mathfrak{a}$ itself. *In other words, $\mathfrak{a}$ (or its T-truncation) is both chaotic and fractal.*

Future Research Directions

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1. Study of the *global spectral operator*

$$\widetilde{\mathfrak{a}} = \xi(\partial),$$

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4. Spectral operator and universality (both for ζ and other L-functions).

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