Part 3: Randomness extraction, asymptotic Hamming distance, and the LIL

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Stochastic immunity

Definition

A set $X$ is **immune** if for each $N \in \mathcal{C}, N \not\subseteq X$. If $\omega \setminus X$ is immune then $X$ is **co-immune**. If $X$ is both immune and co-immune then $X$ is **bi-immune**.

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A set $X$ is **stochastically bi-immune** if for each set $N \in \mathcal{C}, X \upharpoonright N$ satisfies the strong law of large numbers, i.e.,

$$\lim_{n \to \infty} \frac{|X \cap N \cap n|}{|N \cap n|} = \frac{1}{2}.$$
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Could also be called: *weakly Mises-Wald-Church stochastic*
Definition

A sequence $X \in 2^\omega$ is Mises-Wald-Church stochastic if no partial computable monotonic selection rule can select a biased subsequence of $X$, i.e., a subsequence where the relative frequencies of 0s and 1s do not converge to $1/2$. 
Hamming distance $d(\sigma, \tau)$ is given by

$$d(\sigma, \tau) = |\{n : \sigma(n) \neq \tau(n)\}|.$$

Let the collection of all infinite computable sets be denoted by $\mathcal{C}$. Let $p : \omega \to \omega$. For $X, Y \in 2^{\omega}$ and $N \in \mathcal{C}$ we write

$$X \sim_{p,N} Y \iff (\forall \infty n \in N) \ (d(X \mid n, Y \mid n) \leq p(n)).$$

$$X \sim_{p} Y \iff X \sim_{p,\omega} Y.$$  

$$X \prec_{p} Y \iff X \prec_{p,L} Y \ (\exists L \in \mathcal{C}).$$

Easy to understand randomness extraction in terms of $\prec_{p}$.  
Seems hard in terms of $\sim_{p}$. 

Main Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ with $\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{B}) = 1$ be given by

\[
\mathcal{A} = \{ X : X \text{ is weakly 3-random} \}, \\
\mathcal{B} = \{ X : X \text{ is stochastically bi-immune} \}.
\]

Theorem

Let $p : \omega \to \omega$ be any computable function such that $p(n) = \omega^*(\sqrt{n})$. Let $\Phi$ be a Turing reduction. Then

\[
(\forall X \in \mathcal{A})(\exists Y \preceq_p X)(\Phi^Y \notin \mathcal{B}).
\]

Two cases: $\Phi$ maps almost every random to a random, or not.
Question

Why did you obtain results for \( \preceq_p \) but not for \( \sim_p \)? Lack of problem-solving... or a valid reason?
Theorem (Law of the iterated logarithm, Khintchine 1924)

Let $X = (X_0, X_1, \ldots)$ be a random variable on $2^\omega$ having the fair-coin distribution. Let $S_n = \sum_{k=0}^{n-1} X_k = d(X \upharpoonright n, \emptyset \upharpoonright n)$. Then with probability one,

$$\limsup_{n \to \infty} \frac{S_n - \frac{n}{2}}{\varphi(n) \sqrt{n}} = 1,$$

where $\varphi(n) = \sqrt{\frac{1}{2} \log \log n}$. 

• The standard deviation of $S_n$ is simply $\sqrt{n}$, so why the strange $\varphi(n)$?

• Michel Weber (1990): can replace $\varphi(n)$ by arbitrarily slow-growing function if we take $\limsup_{n \in \mathbb{N}}$ for sparse set $\mathbb{N}$. 


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Theorem (Law of the iterated logarithm for subsequences, Michel Weber 1990)

Let \( N = \{\nu_1 < \nu_2 < \cdots \} \subseteq \omega \) and let \( \{Y_n\} \) be an i.i.d. sequence with \( \mathbb{E}(Y_n) = 0 \) and \( \mathbb{E}(Y_n^2) = 1 \). Let \( S_n = Y_1 + \cdots + Y_n \). Let

\[
p_n = |\{m \leq n : N \cap (2^{m-1}, 2^m] \neq \emptyset\}|, \\
\mathcal{L}(k) = \ln p_n \quad \text{if} \quad k \in (2^{n-1}, 2^n].
\]

Then we have

\[
\lim_{j \to \infty} \sup_{j} \frac{S_{\nu_j}}{\sqrt{2\nu_j \mathcal{L}(\nu_j)}} = 1 \quad \text{a.s.}
\]

For \( N = \omega \) we get the usual law of the iterated logarithm. For sparse sets \( N \), the function \( \mathcal{L}(\nu_j) \) is an arbitrarily slow-growing function, so the dominator is standard deviation (\( \sqrt{\nu_j} \)) times a small factor.
Theorem

If $X$ is Kurtz random relative to $A$ and $S_n := d(X \upharpoonright n, A \upharpoonright n)$ then

$$\limsup_{n \to \infty} \frac{S_n - \frac{n}{2}}{\varphi(n) \sqrt{n}} \geq 1.$$
Theorem (K, 2007)

A has non-DNR Turing degree
\[ \implies \text{each Martin-Löf random set } X \text{ is Kurtz random relative to } A. \]

(Deeper but less useful here: the converse also holds; Greenberg and J. Miller, 2009.)

Corollary

A \(\succeq_p X\), where p is computable, X is Martin-Löf random, and
\[ P(\{Y : Y \succeq_p X\}) = 0 \]
\[ \implies A \text{ has DNR degree}. \]
The end