A mathematician reads the Corona–Kim paper
“The Contract Disclosure Mandate and Earnings Management under External Scrutiny”

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Earnings management

Inspector is scrutinizing firms for earnings management
Earnings management = misreporting of earnings
The inspector wants to maximize:

\[ bm^I(F)s - \frac{w}{2}s^2. \]

Here \( m^I(F) \) is the inspector’s conjecture about the level of manipulation by the agent, given the publicly available information, but before deciding on the level of scrutiny \( s \).
Two firms

We want to maximize

\[ f(s_0, \ldots, s_{N-1}) = b(m_0^l(F)s_0 + m_1^l(F)s_1) - \frac{w}{2}(s_0^2 + s_1^2) - \frac{\gamma}{2}(s_0s_1 + s_1s_0). \]

We have

\[ \frac{\partial}{\partial s_i} f = bm_i^l(F) - ws_i - \gamma s_{1-i}. \]

Setting the partial derivatives equal to zero,

\[ bm_0^l(F) - ws_0 - \gamma s_1 = 0, \quad bm_1^l(F) - ws_1 - \gamma s_0 = 0. \]
Matrix formulation

\[
\begin{bmatrix}
w & \gamma \\
\gamma & w
\end{bmatrix}
\begin{bmatrix}
s_0 \\
s_1
\end{bmatrix}
= b
\begin{bmatrix}
m_0(F) \\
m_1(F)
\end{bmatrix}
\]

For \( N = 3 \) (three) firms, we replace \( \gamma \) by \( \gamma/(N - 1) \) and have

\[
bm_0(F) - ws_0 - \frac{\gamma}{2}(s_1 + s_2) = 0.
\]
Projections

This can be written in terms of the identity matrix \( I = [\delta_{ij}] \), the Hadamard identity \( 1 \circ = [1] \) and the projection \( P = \frac{1}{N} 1 \circ \):

\[
Cs = bm^I(\mathcal{F})
\]

where

\[
C = \left( w - \frac{\gamma}{N - 1} \right) I + \frac{\gamma}{N - 1} 1 \circ
\]

\[
= \left( w - \frac{\gamma}{N - 1} \right) I + \frac{\gamma}{N - 1} NP
\]

\[
= \left( w - \frac{\gamma}{N - 1} \right) (I - P) + \left\{ \frac{\gamma N + (N - 1)w - \gamma}{N - 1} \right\} P
\]

\[
= \left( w - \frac{\gamma}{N - 1} \right) (I - P) + \{ w + \gamma \} P
\]
Inverses

Lemma

The inverse of \( x(I - P) + yP \) is \( \frac{1}{x}(I - P) + \frac{1}{y}P \). This is because \( P^2 = P \), \( P = IP = PI \), \( I^2 = I \), and \( P(I - P) = 0 \).
Applying inverses

We can solve for $s$ as follows: by Lemma 1,

\[
C^{-1} = \left\{ \left( w - \frac{\gamma}{N - 1} \right) \right\}^{-1} (I - P) + \{ w + \gamma \}^{-1} P
\]

\[
= \frac{N - 1}{\Gamma - \gamma} I + \left( \frac{-N \gamma}{(w + \gamma)(\Gamma - \gamma)} \right) P
\]

and then

\[
s = C^{-1} bm^l(\mathcal{F})
\]

So

\[
s_i = b \left( \frac{1}{(N - 1)w - \gamma} \left( (N - 1)m^l_i(\mathcal{F}) - \frac{\gamma}{w + \gamma} \sum_j m^l_j(\mathcal{F}) \right) \right)
\]

which agrees with Corona & Kim’s formula (6).
Interior solution: positive definiteness

Lemma

For a subspace $W$,

$$\det(aP_W + bP_{W\perp}) = a^{\dim W} b^{\dim W\perp}$$

and hence

$$\det(aP + b(I - P)) = ab^{N-1}$$

This is true because it is true in some basis.
Verifying positive definiteness

For optimization to make sense we need the Hessian $[\partial^2 f_{s_i s_j}]$ to be negative definite, i.e., $C$ to be positive definite, to have a local maximum of $f$. This occurs iff all the eigenvalues are positive. The characteristic polynomial of $C$ is found as follows:

$$
\lambda I - C = \lambda I - (w - \frac{\gamma}{(N - 1)})I - \frac{\gamma N}{N - 1} P
$$

$$
= \left( \lambda - w + \frac{\gamma}{N - 1} \right) (I - P) + (\lambda - w - \gamma) P
$$

and so

$$
0 \overset{!}{=} \det(\lambda I - C) = (\lambda - w - \gamma) \cdot \left( \lambda - w + \frac{\gamma}{N - 1} \right)^{N-1}
$$

giving $\lambda = w + \gamma$ and $w - \gamma/(N - 1)$. 

Eigenvalues

These are both positive if $w > \gamma/(N - 1) \geq 0$, in particular if $w > \gamma \geq 0$, $N \geq 2$. If $\gamma < 0$ (which Corona and Kim do contemplate) then $w^2 > \gamma^2$, $w > 0$ is exactly what we need to guarantee an interior solution.
More parameters

\( m_i \) occurs in:

\[ r_i = e_i + m_i \]

Reported earnings = earnings + “a bias”

\( \beta_i \) occurs in: The contract (salary) offered by the principal (i.e., shareholders or board) to the CEO is

\[ w_i(r_i) = \alpha_i + \beta_i r_i. \]

\( V_i \) is the payoff of the principal who seeks to maximize \( E[V_i]. \)

\[ V_i = e_i - d_P m_i s_i - w_i(r_i) \]
Verifying (11) and (12)

Let $B = [\beta_j]$, $S = [s_j]$, $M = [m_j]$, and $A = C^{-1}$. We have $S = AM$ and $kM + d_A S = B$, so

$$kM + d_A AM = B$$

where

$$A = b \left( \frac{N - 1}{(N - 1)w - \gamma} (I - P) + \frac{1}{w + \gamma} P \right)$$

and

$$kl + d_A A = kl + d_A b \left( \frac{N - 1}{(N - 1)w - \gamma} (I - P) + \frac{1}{w + \gamma} P \right)$$

$$= \left( k + d_A b \left( \frac{N - 1}{(N - 1)w - \gamma} \right) \right) (I - P) + \left( k + d_A b \frac{1}{w + \gamma} \right) P$$
Let \( \Gamma = (N - 1)w \). Using Lemma 1,

\[
(k + d_A A)^{-1} = \frac{1}{(k + d_A b \left( \frac{N-1}{(N-1)w-\gamma} \right))} (I - P) + \frac{1}{(k + d_A b \frac{1}{w+\gamma})} P
\]

\[
= \left( \frac{\Gamma - \gamma}{(k(\Gamma - \gamma) + d_A b (N - 1))} \right) I
\]

\[
+ \frac{bd_A N\gamma}{(k(w + \gamma) + bd_A)(k(\Gamma - \gamma) + (N - 1)bd_A)} P
\]

And indeed, Corona and Kim's (11) is

\[
m_i = \frac{(k(w + \gamma) + bd_A)((N - 1)w - \gamma)\beta_i + bd_A \gamma \sum_j \beta_j}{(k(w + \gamma) + bd_A)(k((N - 1)w - \gamma) + (N - 1)bd_A)}
\]
Solution with $\beta_1 = \cdots = \beta_N$

It turns out that $\beta_i$ is a function of $\sum_j \beta_j$ and hence all $\beta_i$ are equal, when optimizing $E_P[V_i]$.

We check concavity by

$$\frac{d^2 E_P[V_i]}{d\beta_i^2} < 0.$$  

We allow $\gamma < 0$, but concavity is still verified using

$$\frac{dm_i}{d\beta_i} > 0, \quad \frac{ds_i}{d\beta_i} > 0$$

both of which follow from $w^2 > \gamma^2$, $w > 0$. 