Algebra qualifying exam

University of Hawai‘i at Mānoa

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Instructions:

- You have 4 hours to complete this exam.
- There are 6 sections.
- You should answer 12 questions. Answer at least one question in each of the 6 sections; the first question in each section is often (maybe not always) the easiest. If you have time, you may answer more than 12 questions—your entire paper will be read and taken into consideration.

1. Group theory

In this section, $\mathbf{F}_p$ denotes the finite field with p elements.

1. Let G be a group of order $154 = 2 \cdot 7 \cdot 11$.

(a) Show that G has a subgroup of order 77.

(b) Show that G is not simple.

2. Construct two different non-abelian groups of order 8. Prove that they are not isomorphic.

3. Let p be a prime and let $\mathbf{F}_p$ be the field of order p. Suppose $G \leq \mathrm{GL}_n(\mathbf{F}_p)$ and that $|G| = p^r$ for some r. Show that there is some $x \in \mathbf{F}_p^n$ such that $gx = x$ for every $g \in G$.

4. Provide examples of the following.

(a) A group $G \neq \{e\}$ whose only subgroups are itself and the identity.

(b) A non-solvable, non-simple group G such that all quotients of G are abelian.

(c) A non-cyclic group G all of whose proper subgroups are cyclic.

2. Fields and Galois theory

In this section, $\mathbf{Q}$ denotes the field of rational numbers, C_n denotes the cyclic group of order n, and S_n the symmetric group of degree n.

1. (a) Let $f(x) = x^3 - 3x^2 + 3x + 6 \in \mathbf{Q}[x]$. Show that $f(x)$ is irreducible and show that its Galois group contains a transposition.

(b) What is the splitting field K of $f(x) = (x^2 - 2)(x^2 - 3)$ over $\mathbf{Q}$? What is the Galois group $\mathrm{Gal}(K/\mathbf{Q})$? What are the subfields of K and how are they contained within each other? Don't forget to justify your answers.

(c) Let $K = F(\alpha)$ be an algebraic field extension and suppose $[K : F]$ is odd. Show that $K = F(\alpha^2)$.

2. Provide examples of Galois extensions of fields K/F with the following Galois groups:

(a) C_3

(b) S_3

(c) C_6, with $F = \mathbf{Q}$

3. Consider a tower of field extensions of finite degree:

$$\begin{array}{c} L \\ | \\ K \\ | \\ F \end{array}$$

Is it true that if K/F and L/K are Galois, then so is L/F? If so, prove it. If not, provide a counterexample (with justification).

4. Let $\alpha = \sqrt{5 + \sqrt{5}}$ and let $K = \mathbf{Q}(\alpha)$.

(a) Determine the minimal polynomial $f_\alpha(x)$ of α over $\mathbf{Q}$.

(b) What are the Galois conjugates of α?

(c) What is the discriminant of $f_\alpha(x)$?

(d) Show that $K/\mathbf{Q}$ is Galois and determine its Galois group.

3. Category theory

1. Let **Grp** denote the category of groups and let $\mathcal{F} : \mathbf{Grp} \to \mathbf{Grp}$ send a group G to its opposite group G^{op} (i.e. the group whose underlying set is G, but with the operation $\cdot$ given by $g \cdot h = hg$, where the product hg is the usual product in G).

(a) For a group homomorphism $\varphi : G_1 \to G_2$, what is $\mathcal{F}(\varphi)$? Show that $\mathcal{F}$ is a functor. Is it covariant or contravariant?

(b) Show that $\mathcal{F}$ is naturally isomorphic to the identity functor $\mathrm{id} : \mathbf{Grp} \to \mathbf{Grp}$. (Hint: you may want to first show that the map $G \to G^{\mathrm{op}}$ that sends g to g^{-1} is an isomorphism.)

2. A morphism $\pi : A \to B$ in a category $\mathcal{C}$ is called an *epimorphism* if for every object C in $\mathcal{C}$, the induced function

$$\begin{array}{rcl} \pi^* : \mathrm{Hom}_{\mathcal{C}}(B, C) & \to & \mathrm{Hom}_{\mathcal{C}}(A, C) \\ \varphi & \mapsto & \varphi \circ \pi \end{array}$$

is injective. (This notion is meant to generalize that of a surjective function.)

(a) Show that in the category **Set** of sets, a morphism π is an epimorphism if and only if f is surjective.

(b) Show that for the category **CRing** of commutative rings (with identity, and with ring homomorphisms that map 1 to 1), if R is an integral domain that is not a field and F is its field of fractions, then the natural map $R \to F$ is a non-surjective epimorphism.

3. Let $\mathcal{C}$ be a category and let $f, g : X \to Y$ be two morphisms in $\mathcal{C}$. This is diagrammatically written as

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

A *coequalizer* of f and g is an object Z of $\mathcal{C}$ equipped with a morphism $\pi : Y \to Z$ such that the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] & Z \end{tikzcd}$$

commutes (i.e. $\pi \circ f = \pi \circ g$) and satisfying the following universal property: for every morphism $\pi' : Y \to Z'$ with $\pi' \circ f = \pi' \circ g$, there exists a unique morphism $p : Z \to Z'$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[dr, "\pi'"'] & Z \arrow[d, dashed, "\exists! p"] \\ & & Z' \end{tikzcd}$$

(a) Let **Ab** denote the category of abelian groups. Show that the coequalizer of two homorphisms $f, g : A \to B$ in **Ab** is given by the quotient map

$$\pi : B \to B/\operatorname{im}(f-g)$$

(where $\operatorname{im}(f-g)$ denotes the image of $f-g$), i.e. the quotient map

$$\pi : B \to \operatorname{coker}(f-g).$$

(b) Show that if $\pi : Y \to Z$ is the coequalizer of some pair $f, g : X \to Y$ in some category $\mathcal{C}$, then π is an epimorphism (in the sense of the previous question).

4. Ring theory

In this section, all rings and all ring homomorphisms are unital (i.e. all rings have an identity element and all ring homomorphisms map the identity in the domain to the identity in the target). By an ideal in a ring, we mean a two-sided ideal.

1. (a) Give an example of a commutative ring R together with a non-zero prime ideal in R that is not maximal. Justify.

(b) Give an example of a non-commutative ring R together with a non-zero proper ideal of R. Justify.

2. Recall that a ring R is called *simple* if its only ideals are 0 and R itself.

(a) Show that a commutative simple ring (with $1 \neq 0$) is a field.

(b) Give an example of a non-commutative simple ring (with $1 \neq 0$).

3. A *factorization domain* is an integral domain R such that every non-zero, non-unit $r \in R$ has a factorization into irreducible elements of R (this factorization need not be unique).

(a) Suppose that R is an integral domain in which every (strictly) ascending chain of principal ideals is finite. Show that R is a factorization domain. Conclude that Noetherian integral domains are factorization domains.

(b) Let $R = \{f(x) \in \mathbf{Q}[x] : f(0) \in \mathbf{Z}\}$ be the ring of polynomials over $\mathbf{Q}$ with integral constant coefficient. Show that the prime numbers in $\mathbf{Z}$ are irreducible elements of R. Show that every monomial ax for $a \in \mathbf{Q}^\times$ is reducible. Conclude that R is not Noetherian.

4. (a) Let $R = \mathbf{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} : a, b \in \mathbf{Z}\}$. Show that R is a Euclidean domain.

(b) Let $d \geq 3$ be a squarefree integer and let $R = \mathbf{Z}[\sqrt{-d}]$. Show that 2 is an irreducible element of R, but not a prime element of R. Is R a UFD? (Hint: regarding the primality of 2, consider $-d$ if d is even, and $1 + d$ if d is odd.)

5. Modules and multilinear algebra

In this section, $\mathbf{R}$ denotes the field of real numbers. For a set S, the identity function is denoted $\mathrm{id}_S : S \to S$.

1. Let R be a ring, and let $I, J \subseteq R$ be ideals such that $R = I \oplus J$. Let M be an R-module and $\varphi : M \to I$ be a surjective R-homomorphism. Show that there exists an R-homomorphism $\psi : I \to M$ such that $\varphi \circ \psi = \mathrm{id}_I$.

2. Let $\mathbf{F}_2$ be the field of order 2 and let V be the set of all functions from $\mathbf{F}_2$ to $\mathbf{F}_2$.

(a) Endow V with the "standard" $\mathbf{F}_2$-module structure given as follows: for $\lambda \in \mathbf{F}_2$ and $f \in V$,

$$(\lambda \cdot f)(x) = \lambda \cdot (f(x)).$$

Decompose V into simple F_2-modules.

(b) Now, endow V with the following $\mathbf{F}_2$-module structure: for $\lambda \in \mathbf{F}_2$ and $f \in V$,

$$(\lambda \cdot f)(x) = f(x + \lambda).$$

Show that it can't be decomposed into a direct sum of simple $\mathbf{F}_2$-modules.

3. Let m be a positive integer and let $n = 2m$. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbf{R}^n$. For any i, let $\eta_i \in (\mathbf{R}^n)^*$ be the dual element given by $\eta_i(e_i) = 1$, $\eta_i(e_j) = 0$ for $i \neq j$. Let

$$\omega = \sum_{i=1}^{m} \eta_{2i-1} \wedge \eta_{2i}.$$

(a) There is an $n \times n$ matrix J such that for any $x, y \in \mathbf{R}^n$, $\omega(x, y) = x^T J y$. What is the matrix J? Note: this problems uses the standard identification

$$\wedge^2 (\mathbf{R}^n)^* \cong \mathrm{Alt}_2(\mathbf{R}^n),$$

where $\mathrm{Alt}_2(\mathbf{R}^n)$ is the space of alternating bilinear forms on $\mathbf{R}^n$.

(b) Calculate $\omega^{\wedge m}$ (that is, ω wedged with itself m times).

(c) A $n \times n$ matrix A is called *symplectic* if $A^*\omega = \omega$ (where $A^*\omega$ denotes the pullback of ω by A), i.e. $\omega(Ax, Ay) = \omega(x, y)$ for every $x, y \in \mathbf{R}^n$. Show that if A is symplectic then its determinant is 1.

6. Commutative algebra

In this section, $\mathbf{Q}$ denotes the field of rational numbers and $\mathbf{Z}$ denotes the ring of integers. All rings in this section are commutative with identity, and subrings have the same identity as their ambient ring.

1. Let R be the subring of $\mathbf{Z}[X]$ consisting of all polynomials where the coefficients of X and X^2 are 0.

(a) Show that the field of fractions of R is $\mathbf{Q}(X)$.

(b) Find the integral closure of R.

(c) Given an example of a maximal ideal $\mathfrak{p}$ of $\mathbf{Z}[X]$. Explain why $\mathfrak{p} \cap R$ is also maximal.

2. Let R be a commutative ring, and let M be a non-zero R-module. Show that R has a prime ideal $\mathfrak{p}$ such that $M_{\mathfrak{p}} \neq 0$.

3. Let R be a ring, let M be a noetherian R-module, and let $f : M \to M$ be a surjective homomorphism. Show that f is an isomorphism. (Hint: consider f, $f \circ f$, $f \circ f \circ f, \dots$)