APPLIED MATH QUAL 2017

1. Exam Problems

1.1. **Application of Poincare-Benedixson Theorem.** Consider the system

$$\dot{x} = x - y + 2x^2 + axy - x(x^2 + y^2),$$
$$\dot{y} = y + x + 2xy - ax^2 - y(x^2 + y^2).$$

(1) Show that all trajectories enter or remain in an annular region $r_- \le r \le r_+$, where $r = \sqrt{x^2 + y^2}$ is the radius. Determine the bounds $r_\pm$.
(2) Locate any fixed points.
(3) Find the range of parameter values a for which a periodic orbit exists.

1.2. **Bifurcation problem.** In a simplified laser model the number of excited atoms $N(t)$ and the number of laser photons $n(t)$ are coupled according to the system

$$\dot{n} = GnN - kn,$$
$$\dot{N} = -GnN - fN + p.$$

(1) Briefly describe the meaning of the positive parameters G, k, f and p.
(2) Assuming that N approaches its equilibrium much more rapidly than n and making the approximation $\dot{N} \approx 0$, express $N(t)$ in terms of $n(t)$ and derive a first-order system for n.
(3) Show that $n = 0$ becomes unstable for $p > p_c$ where p_c is to be determined, and deduce the type of bifurcation occurring at the laser threshold p_c.

1.3. **Nature of fixed Points.** Consider the system of differential equations

$$\dot{x} = x(1 - x - y),$$
$$\dot{y} = y(1 - x - y)$$

in the (x, y)-plane.

(1) Show that the fixed points of the system are either at the origin or on a line.
(2) Calculate the Jacobian matrix for a general point (x, y) and deduce that the fixed point at the origin is non-degenerate, whereas the fixed points on the line are not.
(3) Perform linear stability analysis of the fixed point at the origin. Solve the linearized equation.
(4) By dividing the two equations, obtain a new equation of the form $\frac{dx}{dy} = Z(x, y)$. Find general solution to this equation. Explain why your solution is what one would expect given the nature of the fixed point at the origin.

(5) Use the prior results to sketch the phase space diagram. Pay special attention to the behavior of phase space trajectories near the line of fixed points. Does the direction of the trajectories change at this line? Why?

1.4. **Chaos.** The Hénon map is given by:

$$H(x, y) = (1 - ax^2 + y, -bx)$$

where a, b are real parameters.

(1) H is the composition of the transformations H_0, H_1 and H_2 of the point $(x, y) \in \mathbb{R}$

$$H_0(x, y) = (x, 1 + y - ax^2), H_1(x, y) = (bx, y), H_2(x, y) = (y, x).$$

Describe the action of these transformation, and draw pictures to add to your explanation, on the set $E = \{(x, y) \in \mathbb{R}^2; x^2 + (\frac{y}{2})^2 = 1\}$, with $a = 4, b = 2$.

(2) Provide a condition on the parameter b for area-contracting/preserving/expanding property of the map H. *Hint. Compute the absolute value of the determinant of the Jacobian*

(3) Perform linear stability analysis of the fixed points of the Hénon map.

(4) Let $a \neq 0$. Prove that if period doubling bifurcation connected with emerging of period 2 orbit occurs for $H(x, y)$, then it takes place for parameters fulfilling the relation $a = \frac{3}{4}(1 + b)^2$. *Hint. Determine the fixed points and use the necessary condition that an eigenvalue of the Jacobian must be =-1.*

1.5. **PDE problem.** Suppose the human population between the ages of y and $y + \Delta y$ is denoted by $P(t, y)\Delta y$ at any time t, that is, $P(t, y)$ is the population density with respect to y. It is modeled according to the PDE

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial y} + D(t, y)P = 0$$

with boundary condition $P(t, 0) = B(t)$, where $B(t)$ and $D(t, y)$ respectively represent the birth and death rates.

(1) Using the method of characteristics, reduce the PDE to an ODE by making a suitable change of variables, and then find the general solution.

(2) Assuming a constant birth rate B_0 and a time-independent death rate $D(y) = ky$ for some constant k, find the long-term population density $P(y)$ and the total population.

1.6. **Linear Algebra.** Consider the $n \times n$ anti-diagonal matrix $\mathbf{A}$ with complex entries

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & & 0 & a_1 \\ 0 & 0 & & a_2 & 0 \\ \dots & & \ddots & & \\ 0 & a_{n-1} & & 0 & 0 \\ a_n & 0 & & 0 & 0 \end{pmatrix}$$

Under what conditions on $a_1, \cdots, a_n$ is $\mathbf{A}$ unitarily diagonalizable?