APPLIED MATH QUALIFYING EXAM APRIL 2018

Solve all six problems. You have 4 hours. Good luck!
You need to demonstrate proficiency in each area

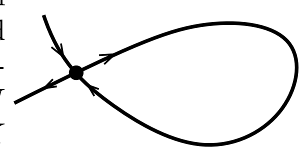
Problem 1: Consider the dynamics of a particle governed by the differential equation

$$\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$$

with parameter $a > 0$.

(1) Find and classify all the fixed points.
(2) Show that the system has a circular limit cycle, and find its amplitude and period.
(3) Determine the stability of the limit cycle.

Problem 2:
Let $\varphi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a flow. Recall that a closed invariant set $A \subset \mathbb{R}^n$ is called an attracting set for φ if there is a neighborhood $U \supset A$ such that $\forall t \geq 0$, $\varphi(t, U) \subset U$ and $\cap_{t>0}\phi(t, U) = A$. Now, consider a set K which is a union of a homoclinic orbit and a hyperbolic equilibrium point, as shown in the figure. Can K be an attracting set?



Problem 3: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field, $G \subset \mathbb{R}^2$ an invariant open set. Show that if there is a function $B : \mathbb{R}^2 \to \mathbb{R}$, $B(x) > 0$ for $x \in G$, such that $\operatorname{div}(B(x)f(x)) = 0$ for $x \in G$, then there is a first integral for f in G, i.e. a function $H : G \to \mathbb{R}$ that is constant along the trajectories: $\dot{H} = \nabla H(x) \cdot f(x) = 0$, $x \in G$.

Hint: Recall that a differential equation $P(x, y)dx + Q(x, y)dy = 0$ *is called exact if* $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, *in which case there is a function* $F(x, y)$ *such that* $\dfrac{\partial F}{\partial x} = P$, $\dfrac{\partial F}{\partial y} = Q$.

Problem 4: Recall that a discrete dynamical system $x \mapsto g(x)$, $g : [a,b] \to [a,b]$, is called chaotic if

(1) g is transitive on $[a,b]$, i.e. for any two subintervals $U_1, U_2 \subset [a,b]$ there is $n > 0$ such that $g^n(U_1) \cap g^n(U_2) \neq \emptyset$.

(2) g has sensitive dependence on the initial conditions, i.e. there is $c > 0$ such that for all $x_0 \in [a,b]$ and any interval $U \ni x_0$, there is $y_0 \in U$ and $n > 0$ such that $|g^n(x_0) - g^n(y_0)| \geq c$.

(3) Periodic points of g are dense in $[a,b]$.

Consider the tent map $g : [0,1] \to [0,1]$ given by

$$g(x) = \begin{cases} 2x, & x \in \left[0, \frac{1}{2}\right] \\ 2x - 1, & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Prove that the dynamical system $x \mapsto g(x)$ is chaotic.

Hint: Find an explicit form of intervals which g^n maps onto $(0,1]$, show that the length of any such interval decreases to zero as $n \to \infty$ and deduce the properties needed for chaos.

Problem 5: The spherical mean of a function u is defined as

$$U = \frac{1}{\alpha(n) n r^{n-1}} \int_{\partial B(x,r)} u \, dS(y),$$

and satisfies

$$\frac{\partial U}{\partial r} = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \Delta u \, dy.$$

Here $\alpha(n)$ is the volume of an n dimensional sphere. You do not need to prove this second relation.

(1) Suppose that $u = u(x)$ is harmonic. Use the formula above to prove the mean value formula for Laplace's equation.

(2) Suppose instead that $u = u(x,t)$ is a solution of the wave equation

$$u_{tt} = \Delta u,$$

show that

$$(r^{n-1} U_r)_r = \frac{1}{\alpha(n)} \int_{\partial B(x,r)} u_{tt} \, dS. \tag{1}$$

(3) Use equation (1) to derive a PDE for function $U(r,t)$.

Problem 6: Let A be an $n \times n$ matrix with integer entries, and assume that A is invertible.

(1) Prove that if A^{-1} has integer entries then $|\det A| = 1$.

(2) Prove that if $|\det A| = 1$ then A^{-1} has integer entries. *Hint: Cramer's rule*