Topology qualifying exam sample

The exam consists of eight questions. You have four hours to complete it. In order to pass, you must exhibit substantial knowledge and ability in each of the following fields: general topology, homotopy theory and covering spaces, and homology theory. Remember to write your test number on the top of this page.

1.

(a) Define the term *open map*, and give an example of an open map, and of a non-open map.

(b) A map $f : X \to Y$ of topological spaces is called *proper* if for every compact $K \subset Y$, $f^{-1}(K) \subset X$ is compact. Give an example of a proper map and of a non-proper map.

(c) Show that every open map is proper.

2. Suppose $Y \subset X$ is a closed, path connected subset of the topological space X, and suppose that the quotient space X/Y is path connected. Show that X is path connected. Given an example that shows that the conclusion need not hold if we don't assume that Y is path connected.

3. Let F_n be the free group of rank n, and let $H < F_n$ be a subgroup of index d. Show that H is also a free group, and calculate its rank.

4. Let $P \subset \mathbb{R}^2$ be a disk with two holes (so P is the space obtained by taking a closed ball in $\mathbb{R}^2$ and removing from it two disjoint open balls). Let X be the space obtained from P by identifying all three boundary components by orientation preserving homeomorphisms. Find a presentation for the fundamental group of P.

5. Construct two non-homotopy equivalent spaces whose fundamental group is $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Carefully state the theorems you are using.

6. Let X be the space obtained by gluing two copies of S^2 along their equatorial S^1 (using the identity map). Calculate the homology groups (with integer coefficients) of X. Call one of the spheres A, and the other B. Write down the long exact sequence of homology groups (with integer coefficients) for the pair (X, A), and calculate every group in this sequence.

7. Describe a CW structure on each of the following spaces, and then calculate its homology groups with integer coefficients (using any method you wish): $S^4 \vee S^2$, $\mathbb{R}P^2$, $S^1 \times S^1 \times S^1$.

8. Suppose $g < h$. Let Σ_g be the surface of genus g, and Σ_h be the surface of genus h. Show that there are no surjective covering maps from Σ_g to Σ_h. Give an example of a surjective map from Σ_h to Σ_g.