Topology qualifying exam sample

The exam consists of eight questions. You have four hours to complete it. In order to pass, you must exhibit substantial knowledge and ability in each of the following fields: general topology, homotopy theory and covering spaces, and homology theory. Remember to write your test number on the top of this page.

1. Let X be a topological space. Show that the intersection of any two dense open sets in X is also dense. Give an example that shows that the conclusion doesn't necessarily hold if the two sets are not required to be open.

2. Let X, Y be topological spaces. Let $f : X \to Y$ be a bijection. Show that f is a homeomorphism if and only if $f(\overline{A}) = \overline{f(A)}$ for every subset $A \subset X$.

3. Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$. How many connected covers does X have of degree 2? How many of degree 3? Explicitly state the theorems you use.

4. Calculate the fundamental group of a Mobius band. Define the term retraction. Explain why there is no retraction from the Mobius band to its boundary circle.

5. Construct a space X whose fundamental group is $\mathbb{Z}/5\mathbb{Z}$. Show that any map from X to S^1 is null-homotopic.

6.

(a) Give an example of a space X and a finite connected cover $\pi : Y \to X$ such that the map $\pi_* : H_1(Y, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is not surjective and not injective.

(b) Given an example of a space X and a finite connected cover $\pi : Y \to X$ such that the map $\pi_* : H_1(Y, \mathbb{Z}) \to H_1(X, \mathbb{Z})$ is surjective and the degree of the cover is not 1.

7.

(a) Let $T^3 = (S^1)^3$ be a three dimensional torus. Calculate $H_i(T^3, \mathbb{Z})$ for every i. Carefully explain your reasoning. You are allowed to use the homology groups of the two dimensional torus without explanation.

(b) Let T^2 be the two dimensional torus. Suppose that $f : T^2 \to T^2$ is a homeomorphism such that $f_* : H_0(T^2, \mathbb{Z}) \to H_0(T^2, \mathbb{Z})$ and $f_* : H_2(T^2, \mathbb{Z}) \to H_2(T^2, \mathbb{Z})$ are both multiplication by 1, and $f_* : H_1(T^2, \mathbb{Z}) \to H_1(T^2, \mathbb{Z})$ is given by multiplication by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Let $M = T^2 \times [0,1]/\sim$ where $\sim$ is the equivalence relation $(x, 0) \sim (f(x), 1)$. Calculate $\dim H_i(M, \mathbb{Q})$ for every i.

(c) Find a presentation for the fundamental group of M.

8. Let X be the topological space obtained from a regular $2n$-gon by identifying opposite edges with parallel orientations. Write a presentation for $\pi_1(X)$, and describe a CW structure on it. Suppose you know that X is a surface. What is its genus?