Applied Math Qualifying Exam

Spring 2025

1. Show that an orthogonal projection P has the property that $\langle Px, x\rangle = \|Px\|^2$.

2. Let M be a normal matrix such that $M^7 = M^6$.

   (a) Prove that M is self adjoint. Give a counterexample of M being self adjoint if M is not normal.

   (b) Prove or disprove that M is a projection matrix.

3. Prove that at any zero-eigenvalue bifurcation in two dimensions, the null-clines always intersect tangentially.

4. Consider the system of ODEs given by

$$\begin{aligned} \dot{x} &= x(1-4x^2-9y^2) - \frac{9}{2}y(2+x), \\ \dot{y} &= y(1-4x^2-9y^2) + 2x(2+x). \end{aligned}$$

   (a) Show that the origin is an unstable fixed point.

   (b) By considering $\dot{V}$, where $V = (1-4x^2-9y^2)^2$, determine what the trajectories approach to in the long-term limit as $t \to \infty$.

5. Assume

$$u(x,t) = v\left(x^2 t^{-1}\right), \quad x \in \mathbb{R},\ t > 0$$

   for some function $v\colon [0,\infty) \to \mathbb{R}$.

   (a) Show

$$u_t = u_{xx}$$

   if and only if

$$4zv''(z) + (2+z)v'(z) = 0 \quad (z \geq 0) \tag{1}$$

   (b) Show that the general solution of Eq. (1) is

$$v(z) = C_1 \int_0^z e^{-s/4} s^{-1/2} ds + C_2$$

(c) Differentiate $v\left(x^2t^{-1}\right)$ with respect to x and select the constant C_1 properly, so as to obtain the fundamental solution.

6. Let $\Omega = \{(x,y)\,|\,4 < x^2+y^2 < 9\}$, $\Gamma_1 = \{(x,y)\,|\,x^2+y^2 = 4\}$, and $\Gamma_2 = \{(x,y)\,|\,x^2+y^2 = 9\}$. Find the solution $u(x,y)$ to

$$\begin{aligned} -\Delta u &= \sqrt{2}, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} &= 0, \quad \text{on } \Gamma_2. \end{aligned}$$

(Recall that in polar coordinates, $\Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$).