# Applied Math Qualifying Exam

Spring 2026

**Problem 1.** *A linear transformation $\mathcal{L} : U \to V$ is said to be independence preserving if $\mathcal{L}(W) \subseteq V$ is linearly independent whenever $W \subseteq U$ is a linearly independent set. Show that $\mathcal{L}$ is independence preserving if and only if it is one-to-one.*

**Problem 2.** *Fix $v$ in an inner-product space $X$ and let $M = \{x \in X : \langle x, v\rangle = 0$. Show that $M$ is a subspace and that it is closed (i.e. every convergent sequence converges inside $M$).*

**Problem 3.** *Consider*

$$-(e^{-x}y')' = \lambda e^{-x}y, \quad 0 < x < 1, \quad y(0) = y(1) = 0.$$

1. *Identify the weight function $w(x)$ and verify the Sturm-Liouville form.*
2. *Find the first eigenvalue and eigenfunction (analytically or numerically).*
3. *Verify the orthogonality of eigenfunctions with respect to $w(x)$.*

**Problem 4.** *Consider the inhomogeneous problem*

$$L[\phi] \equiv \frac{d^2\phi}{dx^2} = f(x)$$

*subject to homogeneous boundary conditions* $\phi = 0$ *at* $x = 0$ *and* $x = 1$.

1. *Find the Green's function* $G(x, x_0)$ *using the related eigenvalue problem* $L[\phi] = -\lambda\phi$*, subject to the same boundary conditions.*

2. *Find another representation of the Green's function by considering the related problem* $L[G(x, x_0)] = \delta(x - x_0)$*, where* $\delta(x - x_0)$ *is the Dirac delta function.*

**Problem 5.** *Consider a $C^1$ system on $\mathbb{R}^n$:*

$$\dot{x} = Ax + G(x),$$

*where $A \in \mathbb{R}^{n\times n}$ is hyperbolic, $G(0) = 0$, and $DG(0) = 0$.*

1. *State the Local Stable Manifold Theorem and define $W^s_{\mathrm{loc}}(0)$ and $W^u_{\mathrm{loc}}(0)$.*
2. *For $n = 2$ with eigenvalues $\lambda_1 < 0 < \lambda_2$, describe how to compute the local stable and unstable manifolds as graphs over the corresponding eigenspaces.*
3. *Apply this to the planar system*

$$\begin{cases} \dot{x} = x(ax + by + c), \\ \dot{y} = y(dx + ey + f), \end{cases}$$

*and determine whether local stable/unstable manifolds exist, and if so, their tangent directions at the origin.*

**Problem 6.** *Consider the diffusion equation*

$$u_t = u_{xx}$$

*on the interval* $0 < x < \ell$*, subject to the boundary conditions* $u(0,t) = u(\ell,t) = 0$ *and the initial condition* $u(x,0) = 1$*.*

1. *Write the initial condition as a Fourier sine series.*
2. *Solve the problem using the method of separation of variables.*