Applied Math Qualifying Exam

Fall 2024

1. Let P be a projection operator on a real inner produce space U. Prove that the operator $I - 2P$ is an isometry if and only if P is self adjoint.

2. Two matrices $M, N \in \mathbb{R}^{n\times n}$ are called simultaneously diagonalizable if there exists an invertible matrix D such that $D^{-1}MD$ and $D^{-1}ND$ are both diagonal.

   (a) Prove that if M and N are simultaneously diagonalizable then they commute.

   (b) Prove that if $MN = NM$ and if one of the matrices has n distinct eigenvalues then M and N are simultaneously diagonalizable.

3. The ODE system characterizing the change of concentration of chemical species S_1 and S_2 in a *chemostat* model is as follows:

$$\begin{aligned} \dot{x}_1 =& k_1 x_2(k^* - x_1) + \alpha_1 - \beta_1 x_1, \\ \dot{x}_2 =& -k_1 x_2(k^* - x_1) + \alpha_2 - \beta_2 x_2, \end{aligned} \tag{1}$$

   where k_1, k_2, α_1, α_2, β_1, β_2 are positive *reaction rate constants*, $x_1(t)$ and $x_2(t)$ are the concentration of species S_1 and S_2 at time $t \geq 0$, respectively, and $k^* = k_2/k_1$. We abbreviate Eq. (1) by $\dot{x} = F(x)$. For simplicity, you are allowed to set $\alpha_1 = \alpha_2 = k_1 = k_2 = 1$, $\beta_1 = 2$, and $\beta_2 = 3$.

   (i) Prove that there exists a unique positive equilibrium $P = (x_1^*, x_2^*)$ (the so-called *positive steady state*) for the ODE (1). Moreover, show (x_1^*, x_2^*) is locally asymptotically stable.

   (ii) Show there exists no periodic solution. Hint: Use Dulac criterion: Construct a $\mathcal{C}^1$ function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ and show that

$$\mathrm{div}\nabla(\varphi F) \coloneqq \frac{\partial(\varphi F_1)}{\partial x_1} + \frac{\partial(\varphi F_2)}{\partial x_2}$$

   has the same sign almost everywhere in a simply connected region of $\mathbb{R}^2$, where $F = (F_1, F_2)$.

4. Consider the second-order ODE

$$\ddot{x} + \frac{1}{2}\begin{bmatrix} \omega^2 + 1 & \omega^2 - 1 \\ \omega^2 - 1 & \omega^2 + 1 \end{bmatrix} x = 0, \tag{2}$$

   where $\omega > 0$ is a given real number.

(i) Show that every solution $x\colon \mathbb{R}^2 \to \mathbb{R}^2$ of Eq. (2) is bounded.

(ii) Prove that every solution of Eq. (2) is periodic if and only if ω is rational.

5. Consider the operator $L = -\frac{d}{dx}$ on the space of C^2 functions satisfying periodic boundary conditions:

$$u(-1) = u(1), \qquad u'(-1) = u'(1).$$

(a) Compute the (formal) adjoint of L with respect to the inner product

$$\langle u, v\rangle = \int_{-1}^{1} uv\, x^2 dx.$$

(b) Show that the operator $S = L^*L$ is self-adjoint and positive. Is it definite?

(c) Recall that the Rayleigh quotient for a self-adjoint linear operator S is defined as

$$R[u] = \frac{\langle u, Su\rangle}{||u||^2}.$$

Show that if $v \neq 0$ minimizes $R[u]$ then (i) v is an eigenvector of S and (ii) its eigenvalue is $\lambda = min\{R[u],\, u \neq 0\}$.

(d) For the operator S given in part (c) find the minimum value of the Rayleigh quotient and an associated minimizing function.

6. Let U be an open bounded subset of $\mathbb{R}^n$ and $v \in C^2(\overline{U})$. We say v is subharmonic if

$$-\Delta v \leq 0 \qquad \text{in } U.$$

(a) For subharmonic v prove that

$$v(x) \leq \frac{1}{\alpha(n) r^n} \int_{B(x,r)} v(y) dy$$

for all $B(x, r) \subset U$. Here $B(x, r)$ is the closed ball around x of radius r and recall that $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

(b) Use part (a) to state and prove a max and/or min theorem for subharmonic functions.

(c) Show that if u is harmonic then $v = |\nabla u|^2$ is subharmonic.