Applied math qualifying exam
Fall 2023

1. Let A be a square matrix satisfying $A^2 = A$. Show that

$$\text{Null}(A^T) = \text{Range}(A - I)$$

if and only if $A = A^T$.

2. Let A be a compact, self-adjoint, positive definite linear operator on an infinite-dimensional Hilbert space H. For each finite-dimensional subspace $V \subset H$, let $A_V : V \to V$ denote the map that sends $x \in V$ to the orthogonal projection of Ax onto V. Use induction to show that for each positive integer k,

$$\sup_{\substack{V \subset H, \\ \dim V = k}} \text{Tr}(A_V) = \lambda_1 + \lambda_2 + \cdots + \lambda_k,$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ are the eigenvalues of A and Tr denotes the trace of a linear transformation.

3. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ be smooth functions that vanish at the origin. Let $A = DF(0)$ and $B = DG(0)$. Show that if $B = -A^T$, then the dynamical system

$$\begin{aligned} \dot{x} &= -\varepsilon x + F(y), \\ \dot{y} &= -\varepsilon y + G(x) \end{aligned}$$

has a stable equilibrium point at the origin whenever $\varepsilon > 0$. In the case where A is diagonalizable and $B = -A^T$, find a formula for the solution $(x(t), y(t))$ to the linearized system at the origin in terms of the eigenvalues and eigenvectors of A.

4. Consider the dynamical system

$$\begin{aligned} \dot{x} &= x(1 - x - y), \\ \dot{y} &= y\left(x - \frac{1}{2}\right). \end{aligned}$$

   (a) Show that the first quadrant $(x > 0, y > 0)$ is an invariant set.

   (b) Find all equilibrium solutions and perform their linear stability analysis.

   (c) Use the Lyapunov function $F(x, y) = 2x + 2y - 2 - \log(2x) - \log(2y)$ to show that all solutions in the first quadrant $(x > 0, y > 0)$ approach a single point as $t \to \infty$.

5. Find a function $u(x)$ that minimizes

$$\frac{1}{2}\int_0^1 e^x u'(x)^2\, dx - \int_0^1 e^{2x} u(x)\, dx$$

among all functions in $C^2([0, 1])$ that vanish at $x = 0$ and $x = 1$.

6. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with smooth boundary. Let $f \in C^\infty(\overline{\Omega})$, and let α be a positive constant. Show that if α is sufficiently small and if the problem

$$\begin{aligned} -\Delta u + u &= f, && \text{in } \Omega, \\ \frac{\partial u}{\partial n} - \alpha u &= 0, && \text{on } \partial\Omega \end{aligned}$$

has a smooth solution, then it is unique. You can use the following result without proof: There exists a positive constant C such that for all $v \in C^1(\overline{\Omega})$, $\|v\|^2_{L^2(\partial\Omega)} \leq C(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)})$.