Applied math qualifying exam
Spring 2024

1. A matrix $A \in \mathbb{C}^{n\times n}$ is called nilpotent if $A^k = 0$ for some positive integer k. Show that A is nilpotent if and only if every eigenvalue of A is zero.

2. Let U be a closed subspace of a Hilbert space V, and let $U^\perp$ denote its orthogonal complement. Let P be the orthogonal projector onto U, and let $T : V \to V$ be a linear transformation. Show that both U and $U^\perp$ are invariant subspaces of T (i.e. $T(U) \subseteq U$ and $T(U^\perp) \subseteq U^\perp$) if and only if $PT = TP$.

3. Consider the dynamical system

$$\begin{aligned} \dot{x} &= a - x + x^2 y, \\ \dot{y} &= 1 - x^2 y. \end{aligned}$$

   (a) Verify that the system has an equilibrium solution at $\left(a+1, \frac{1}{(a+1)^2}\right)$ whenever $a \neq -1$, and compute the linearized system around this equilibrium solution.

   (b) Show that as a varies, a bifurcation occurs at $a = 0$. Describe this bifurcation.

4. Let $A(t)$ be a real $n \times n$ matrix whose entries depend smoothly on t, and consider the dynamical system

$$\dot{x} = A(t)x$$

   on $\mathbb{R}^n$. Show that if $\int_0^\infty \|A(t)\|\, dt < \infty$, then for any initial condition, $\lim_{t\to\infty} x(t)$ exists and is finite. (Hint: Start by proving an estimate for $\|x(t)\|$.)

5. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with smooth boundary, and let $T > 0$. Let $a \in C^\infty(\overline{\Omega})$ be a positive function, and let $g \in C^\infty(\partial\Omega)$ and $u_0, w_0 \in C^\infty(\overline{\Omega})$. Show that if the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \operatorname{div}(a\nabla u) && \text{in } \Omega \times (0,T), \\ u &= g && \text{on } \partial\Omega \times (0,T), \\ u &= u_0 && \text{on } \Omega \times \{0\}, \\ \frac{\partial u}{\partial t} &= w_0 && \text{on } \Omega \times \{0\} \end{aligned}$$

   has a smooth solution, then it is unique. (Hint: Consider the energy $E(t) = \frac{1}{2}\int_\Omega \left(\frac{\partial v}{\partial t}\right)^2 + a|\nabla v|^2\, dx$ of the difference $v = u - \widetilde{u}$ between two solutions u and $\widetilde{u}$.)

6. Consider the inviscid Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

(a) Show that if α is a positive number and u is a solution of the form $u(x,t) = t^{\alpha-1}w\left(\frac{x}{t^\alpha}\right)$, then the function w must satisfy

$$(w - \alpha y)\frac{dw}{dy} + (\alpha - 1)w = 0,$$

where $y = \frac{x}{t^\alpha}$.

(b) Assume $\alpha \neq 1$. By re-expressing the above equation in terms of $\frac{dy}{dw}$, solve for $y = y(w)$ to deduce an implicit formula for u.