Sample qualifying exam 1

1. Let U, V, and W be finite-dimensional inner product spaces, and suppose $R : U \to V$ and $S : V \to W$ are both linear and $\text{Range}(R) = \text{Null}(S)$. Show that the map $T := RR^* + S^*S$ is invertible.

2. Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices satisfying $AB = BA$. Assume that A and B each have n distinct eigenvalues.

   (a) Show that there exists an orthogonal matrix Q such that both Q^TAQ and Q^TBQ are diagonal.

   (b) Use part (a) to show that $e^{A+B} = e^Ae^B$.

3. Let $\mu > 0$. Consider the following system in $\mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= \mu x - y - x(x^2 + y^2) \\ \dot{y} &= x + \mu y - y(x^2 + y^2). \end{aligned}$$

   (a) Show that this system has a periodic orbit given in polar coordinates by $\phi_t(\sqrt{\mu}, \theta_0)$, where ϕ_t denotes the flow of the system.

   (b) Verify that $\Sigma = \{(r, \theta) \in \mathbb{R} \times S^1 \mid r > 0, \theta = \theta_0\}$ is a cross-section for the flow, and find the Poincaré map $P : \Sigma \to \Sigma$. Show that it has an asymptotically stable fixed point at $r = \sqrt{\mu}$.

4. Show that the problem

$$\begin{aligned} -t\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial t} &= 0, && x \in \mathbb{R},\ t > 0, \\ u(x, 0) &= 3x, && x \in \mathbb{R} \end{aligned}$$

   has no C^1 solution.

5. Let $V = \{v \in C^2([0, 1]) \mid v(0) = 0,\ v(1) = v'(1)\}$. For $v \in V$, let $Lv = -v''$. Find the eigenvalues and eigenfunctions of L.

6. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose that u is a smooth function satisfying

$$\begin{aligned} -\Delta u &= f, && \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= g, && \text{on } \partial\Omega, \end{aligned}$$

   where $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are given functions satisfying $\int_\Omega f\,dx + \int_{\partial\Omega} g\,dS = 0$. Show that for every smooth function v, we have

$$E(u) \le E(v)$$

   where

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega fv\,dx - \int_{\partial\Omega} gv\,dS.$$

   Is the inequality strict for $v \neq u$?

Sample qualifying exam 2

1. Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Show that a vector $u \in \mathbb{R}^n$ is an eigenvector of A if and only if u is a critical point of the map $F : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ given by $F(x) = \frac{x^T Ax}{x^T x}$ (i.e. $\frac{\partial F}{\partial x_i}$ vanishes at $x = u$ for each $i = 1, 2, \ldots, n$).

2. Let $A \in \mathbb{C}^{n\times n}$ be Hermitian and unitary. Show that there exists a (possibly trivial) subspace $V \subseteq \mathbb{C}^n$ such that for every $x \in V$ and every $y \in V^\perp$, $A(x + y) = x - y$. (Recall that A is Hermitian if and only if A equals its conjugate transpose.)

3. Consider the following variation of the predator-prey system:

$$\begin{aligned}\dot{x} &= x(x(1-x) - y),\\ \dot{y} &= y(x - a),\end{aligned}$$

   where $x, y \geq 0$ represent prey and predator populations, respectively, and $a > 0$ is a parameter.

   (a) Find the equilibrium points, perform linear stability analysis, and classify the equilibrium points accordingly.

   (b) Deduce that predators go extinct if $a > 1$.

   (c) Show that a Hopf bifurcation occurs at $a = \frac{1}{2}$.

4. Consider the following dynamical system on the positive orthant $\{x \in \mathbb{R}^n \mid x_i > 0,\ i = 1, 2, \ldots, n\}$:

$$\begin{aligned}\dot{x}_1 &= x_1(r_1 - a_{11}x_1 - a_{12}x_2),\\ \dot{x}_2 &= x_2(-r_2 + a_{21}x_1 - a_{22}x_2 - a_{23}x_3),\\ &\vdots\\ \dot{x}_i &= x_i(-r_i + a_{i,i-1}x_{i-1} - a_{ii}x_i - a_{i,i+1}x_{i+1}),\\ &\vdots\\ \dot{x}_n &= x_n(-r_n + a_{n,n-1}x_{n-1} - a_{nn}x_n).\end{aligned}$$

   Assume that the scalars r_j and a_{ij} appearing above are all positive. Prove that if the system above has an equilibrium point (in the positive orthant), then this point is globally stable. (Hint: Look for a Lyapunov function of the form $\sum_i \alpha_i(x_i - \beta_i \log x_i)$.)

5. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $T > 0$. Show that if u is a smooth solution of

$$\begin{aligned}\frac{\partial u}{\partial t} - \Delta u &= 0, && \text{in } \Omega \times (0, T),\\ u &= 0, && \text{on } \partial\Omega \times [0, T],\\ u &= u_0, && \text{on } \Omega \times \{0\},\end{aligned}$$

then

$$\int_\Omega u(x,t)^2\,dx \le e^{-2Ct}\int_\Omega u_0(x)^2\,dx$$

for every $t \in [0,T]$, where

$$C = \inf\left\{ \frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Omega v^2\,dx} \mid v \in C^1(\overline{\Omega}),\ v = 0 \text{ on } \partial\Omega, v \not\equiv 0 \text{ in } \Omega \right\}.$$

What is the value of C when $n = 1$ and $\Omega = (0,1)$?

6. Let $\Omega = \{(x,y) \mid 1 < x^2 + y^2 < 4\}$, $\Gamma_1 = \{(x,y) \mid x^2 + y^2 = 1\}$, and $\Gamma_2 = \{(x,y) \mid x^2 + y^2 = 4\}$. Find the solution $u(x,y)$ to

$$\begin{aligned} -\Delta u &= 1, && \text{in } \Omega, \\ u &= 0, && \text{on } \Gamma_1, \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \Gamma_2. \end{aligned}$$

(Recall that in polar coordinates, $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.)

Sample qualifying exam 3

1. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H with inner product $(\cdot,\cdot)$. Let $A : H \to H$ be the linear operator

$$Ax = \sum_{n=1}^{\infty} (x, e_{n+1}) e_n.$$

   (a) Is A self-adjoint?

   (b) Is A compact?

   (c) What is the norm of A?

   (d) Find the eigenvalues and eigenvectors of A.

2. Consider the differential operator

$$K = -\frac{d^2}{dt^2} + \frac{t^2}{4}$$

   acting on $C^\infty((-\infty, \infty))$. Show that if $\lambda \in \mathbb{R}$ is an eigenvalue of K, then so are $\lambda + 1$ and $\lambda - 1$. (Hint: Let $A = \frac{t}{2} + \frac{d}{dt}$, and $B = \frac{t}{2} - \frac{d}{dt}$, and consider $K - AB$ and $AB - BA$.)

3. Let $A(t)$ be an $n \times n$ real matrix whose entries depend smoothly on t. Suppose that there exists a skew-symmetric matrix $K(t)$ such that

$$\dot{A}(t) = A(t)K(t)$$

   for all t. Show that if $A(0)$ is orthogonal, then $A(t)$ is orthogonal for all t.

4. Consider the planar dynamical system given in polar coordinates by

$$\begin{aligned} \dot{r} &= \alpha r - r^3, \\ \dot{\theta} &= \sin\theta + \beta. \end{aligned}$$

   (a) Find conditions on the parameters α and β under which the system has: one equilibrium point and no periodic orbits, two equilibrium points and no periodic orbits, no equilibrium points and one periodic orbit. Sketch the phase portrait for each of the cases.

   (b) Show that for a fixed $|\beta| > 1$ the system undergoes a Hopf bifurcation at $\alpha = 0$.

   (c) Describe bifurcations that occur for fixed $\alpha > 0$ when β is varied. Sketch the corresponding bifurcation diagram.

5. Use Duhamel's principle to solve the following partial differential equation:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) + t\sin x, \quad 0 < x < \pi,\ t > 0, \\ u(x,0) &= 0, \qquad\qquad 0 < x < \pi, \\ u(0,t) &= 0, \qquad\qquad t > 0, \\ u(\pi,t) &= 0, \qquad\qquad t > 0. \end{aligned}$$

6. Suppose that u is a smooth solution of the initial-value problem

$$\frac{\partial u}{\partial t} - a\Delta u + b|\nabla u|^2 = 0, \qquad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1}$$

$$u = g, \qquad \text{on } \mathbb{R}^n \times \{0\}, \tag{2}$$

where $a > 0$, $b \in \mathbb{R}$, and $g : \mathbb{R}^n \to \mathbb{R}$ is given.

(a) Show that if $\phi : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying

$$a\phi'' + b\phi' = 0,$$

then the function $w = \phi(u)$ satisfies

$$\frac{\partial w}{\partial t} - a\Delta w = 0, \qquad \text{in } \mathbb{R}^n \times (0,\infty),$$

$$w = \phi(g), \qquad \text{on } \mathbb{R}^n \times \{0\}.$$

(b) Use part (a) to find an explicit formula for the solution to (1-2) in terms the heat kernel $\frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)}$.

Sample qualifying exam 4

1. Find a function $u : (0, \pi) \to \mathbb{R}$ satisfying

$$u(x) + \int_0^\pi \sin(x - y)u(y)\, dy = \cos x.$$

(Hint: $\sin(x - y) = \sin x \cos y - \cos x \sin y$.)

2. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal sequence in a Hilbert space H with inner product $(\cdot, \cdot)$, and let $\{\lambda_n\}_{n=1}^\infty$ be a bounded sequence in $\mathbb{R}$. Let $A : H \to H$ be the linear operator $Ax = \sum_{n=1}^\infty \lambda_n (x, e_n) e_n$.

   (a) Show that A is bounded.

   (b) Show that if $\lambda_n \not\to 0$, then A is not compact.

3. Let A be an $n \times n$ matrix whose eigenvalues all have negative real part, and let $b(t)$ be a vector in $\mathbb{R}^n$ that depends smoothly on t. Show that if $y(t)$ solves the initial value problem

$$\dot{y}(t) = Ay(t) + b(t), \quad y(0) = y_0,$$

then there exist constants $C_1, C_2 > 0$ such that

$$\|y(t)\| \le C_1 \left(e^{-C_2 t} \|y_0\| + \int_0^t e^{-C_2(t-s)} \|b(s)\|\, ds \right).$$

4. Consider the following system:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= y - x^3. \end{aligned}$$

   (a) Show that origin is a fixed point, and that linear stability analysis does not yield an answer regarding its stability.

   (b) Use a Lyapunov function of the form $V(x, y) = ax^4 + bx^2 + cxy + dy^2$ to deduce that the origin is unstable.

5. Solve the initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{5}{6} \frac{\partial u}{\partial x} - 6x, && x \in \mathbb{R},\ t > 0, \\ u(x, 0) &= 12x, && x \in \mathbb{R}. \end{aligned}$$

6. Let $\Omega = \{x \in \mathbb{R}^3 \mid |x| < 1\}$, and let $u(x, t)$ be a smooth solution of the wave equation

$$\begin{aligned} u_{tt} &= \Delta u, && \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial n} &= 0, && \text{on } \partial\Omega \times (0, \infty). \end{aligned}$$

For each $r \in (0, 1)$ and $t \in (0, \infty)$, let $U(r, t)$ denote the average value of $u(\cdot, t)$ over a sphere of radius r. Show that

$$\int_0^1 r^2 U_{tt}(r, t)\, dr = 0$$

for all t.