Math 372 Problems #2

1 Homework

This section contains the assigned homework, due on 1/21/2019.

Problem 1 *Prove that there is no uniform distribution on the sample space* $\Omega = \{0, 1, 2, 3, \ldots\} = \mathbb{N}$. *(Recall: if* m *is a uniform distribution on a sample space* Ω*, then there is a fixed* $0 \leq a \leq 1$ *such that* $m(\omega) = a$ *for all* $\omega \in \Omega$*.)*

Solution Suppose m is a constant function such that $m(n) = a$ for all $n \in \mathbb{N}$. If $a = 0$, then $\sum_{n=0}^{\infty} m(n) = 0$, thus, m is not a distribution. If $a > 0$, since $\sum_{n=0}^{\infty} m(n) = \sum_{n=0}^{\infty} a$ diverges, m cannot be a distribution.

□

2 Recommended Problems

These are the recommended problems. Two of them will appear on the quiz on 1/21/2019.

Problem 2 *Suppose that* Ω *is a finite sample space and there is a distribution defined on* Ω*. Prove that if* A *and* B *are events over the sample space* Ω *with* $A \subseteq B$*, then* $P(A) \leq P(B)$*. Is it true that if* $A \subsetneq B$*, then* $P(A) < P(B)$*? For the last question, either prove it is true or present a counterexample.*

Solution Suppse $A = \{\omega_1, \ldots, \omega_k\}$ and $B = \{\omega_1, \ldots, \omega_k, \ldots, \omega_{k+n}\}$. Given a distribution function, m, we have that

$$P(A) = m(\omega_1) + \cdots + m(\omega_k) \leq m(\omega_1) + \cdots + m(\omega_k) + \cdots + m(\omega_{n+k}) = P(B)$$

Thus, $P(A) \leq P(B)$.

Consider the sample space $\Omega = \{0, 1\}$ with distribution m such that $m(0) = 1$ and $m(1) = 0$. Let $A = \{0\}$ and $B = \{0, 1\}$. Although $A \subsetneq B$, note that $P(A) = m(0) = 1 = 1 + 0 = m(0) + m(1) = P(B)$. Thus, $A \subsetneq B$ does not imply that $P(A) < P(B)$.

□

Problem 3 *Consider a well-shuffled standard deck of 52 cards. Let* S *be the event "the top card is a spade",* F *is the event "the top card is a face card" and* A *is the event "the top card is an ace".*

1. *Find* $P(S)$*,* $P(F)$ *and* $P(A)$*.*

2. *Find* $P(F \cup A)$, $P(S \cup F)$ *and* $P(S \cup A)$.

3. *Find* $P(F \cap A)$, $P(S \cap F)$ *and* $P(S \cap A)$.

4. *Find* $P(F \setminus S)$ *and* $P(S \setminus F)$.

Solution

1. $P(S) = 1/4$, $P(F) = 12/52 = 3/13$ and $P(A) = 4/52 = 1/13$.

2. $P(F \cup A) = (12+4)/52 = 4/13$, $P(S \cup F) = (10+3+9)/52 = 22/52 = 11/26$ and $P(S \cup A) = (12+1+3)/52 = 16/52 = 4/13$.

3. $P(F \cap A) = 0/52 = 0$, $P(S \cap F) = 3/52$ and $P(S \cap A) = 1/52$.

4. $P(F \setminus S) = (12-3)/52 = 9/52$ and $P(S \setminus F) = (13-3)/52 = 10/52$.

□

Problem 4 *Suppose that* $\Omega = \{a, b, c\}$ *and* m *is a distribution on* Ω *such that* $m(a) = 1/4$ *and* $m(b) = 1/4$. *Find the probabilities of all 8 different subsets of* Ω.

Solution Since $m(a) = 1/4$ and $m(b) = 1/4$, we know that $m(c) = 1/2$. Thus, $P(\emptyset) = 0, P(\{a\}) = \frac{1}{4}, P(\{b\}) = \frac{1}{4}, P(\{c\}) = \frac{1}{2}, P(\{a,b\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, P(\{a,c\}) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}, P(\{b,c\}) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ and $P(\{a,b,c\}) = 1$

□

Problem 5 *Suppose a fair 6-sided die is rolled 3 times. What is the probability that the first roll is divisible by 2, the second roll is divisible by 3 and the third roll is divisible by 4?*

Solution Let A be the event that the first roll is divisible by 2, B be the event that the second roll is divisible by 3 and let C be the event that the third roll is divisible by 4. Since $\{2, 4, 6\}$ are divisible by 2, $P(A) = 1/2$. Since $\{3, 6\}$ are divisible by 3, $P(B) = 1/3$. Since $\{4\}$ is divisible by 4, $P(C) = 1/6$. So the probability that the first roll is divisible by 2, the second roll is divisible by 3 and the third roll is divisible by 4 is $(1/2)(1/3)(1/6) = 1/36$.

□

Problem 6 *Suppose a fair coin is flipped 5 times. What is the probability that heads will come up exactly twice? (A tree diagram may help.)*

Solution There are ten possiblities, thus, the probability is $10/32 = 5/16$.

□

Problem 7 *Suppose that* 10% *of the fish in a school are blue,* 25% *are in the family* Acantharus *and* 30% *are either blue or in the family* Acantharus. *What is the probability that a randomly selected fish is both blue and in the family* Acantharus*?*

Solution Let B be the event that the randomly selected fish is blue and let A be the event that the randomly selected fish is in the family *Acantharus*. $P(B) = 0.1$, $P(A) = 0.25$ and $P(B \cup A) = 0.3$. Since

$$0.3 = P(B \cup A) = P(B) + P(A) - P(B \cap A) = 0.1 + 0.25 - P(B \cap A)$$

we have that $P(B \cap A) = 0.05$.

□

Problem 8 *For each of problems 5, 6 and 7, state your answer using odds instead of probabilities.*

Solution For problem 5, the odds are 1 to 35. For problem 6, the odds are 5 to 11. For problem 7, the odds are 1 to 19.

□

Problem 9 *Suppose that* $P(A) = P(B)$, $P(A \cup B) = p$ *and* $P(A \cap B) = q$. *What is* $P(A)$*?*

Solution Since $p = P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(A) - P(A \cap B) = 2P(A) - q$, we have that $P(A) = {}^1\!/\!_2(p + q)$.

□

Problem 10 *Define a distribution,* m*, on the sample space* $\Omega = \{a, b, c, d\}$ *such that*

1. $P(\{a, b, c\}) = {}^3\!/\!_4$
2. $P(\{a, b, d\} = {}^1\!/\!_2$
3. $P(\{c\}) = {}^1\!/\!_2$.

Solution Define m so that $m(a) = 0, m(b) = {}^1\!/\!_4, m(c) = {}^1\!/\!_2$ and $m(d) = {}^1\!/\!_4$.

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