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**Definitions**

Let $f$ be a function and $I$ be an interval. $f$ is *concave up* on $I$ iff its derivative $f'$ is increasing on $I$.

$f$ is *concave down* on $I$ iff its derivative $f'$ is decreasing on $I$.

$\langle a, f(a) \rangle$ is an inflection point of $f$ iff there is a change in concavity from up to down or from down to up at $a$.

So if a function is increasing and concave up on an interval, then the slopes of the tangents are positive and getting steeper. On the other hand, if a function is increasing and concave down on an interval, then the slopes of the tangents are positive but getting less steep. Etc. The four different possibilities are pictured below.
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**Theorem**

*Suppose the function $f$ is continuous on the interval $I$ and differentiable on its interior $I^\circ$.*

1. If $f'(x) = 0$ for all $x$ in $I^\circ$, then $f$ is constant on $I$.
2. If $f'(x) > 0$ for all $x$ in $I^\circ$, then $f$ is increasing on $I$.
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If we apply this theorem to $f'$ and $f''$ instead of $f$ and $f'$, we obtain results about concavity.

**Corollary**

*Suppose $f'$ is continuous on the interval I and differentiable on its interior $I°$.*

(1) If $f''(x) > 0$ for all $x$ in $I°$, then $f$ is concave up on $I$.
(2) If $f''(x) < 0$ for all $x$ in $I°$, then $f$ is concave down on $I$.
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**Proposition**

If $\langle a, f(a) \rangle$ is an inflection point of $f$, then either $f''(a) = 0$ or $f''(a)$ does not exist.
Warning

It does not go the other way. It can happen that $f''(a) = 0$ even though $\langle a, f(a) \rangle$ is not an inflection point of $f$.

Let $f(x) = x^4$. Then $f'(x) = 4x^3$ which is a polynomial and continuous everywhere. Also, $f''(x) = 12x^2$. So $f''(0) = 0$, but $f''(x) > 0$ if $x \neq 0$. So $f''(x) > 0$ on $(-\infty, 0)$ and on $(0, +\infty)$. Then Corollary 2 implies $f$ is concave up on $(-\infty, 0]$ and on $[0, +\infty)$. Thus $f$ is concave up on $(-\infty, +\infty)$. So $f$ has no inflection points at all even though $f''(0) = 0$. The graph is just very flat near 0, but there is no concavity switch and no inflection point.
Now we will analyze and sketch some functions using the tools at hand. What we need to know in each case is where \( f \) and \( f' \) are continuous and where \( f' \) and \( f'' \) are positive and negative.
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Problem. $f(x) = x^3 - 3x + 2$. Find where $f$ is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch $f$ showing all these things.
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The graph follows. The local extrema are indicated in blue and the inflection point in green.
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Because \( f \) is decreasing on \((-\infty, -1]\) and increasing on \([-1, +\infty)\), we see that \( f(-1) = -1 \) is a local (and absolute) minimum value. There are no local maxima.
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\[ f''(x) = 36x(x + \frac{2}{3}) > 0; \]
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\[ f''(x) = 36x(x + \frac{2}{3}) > 0; \text{ i.e., } f''(x) > 0 \text{ on } (-\infty, -\frac{2}{3}). \]
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Corollary 2 implies that \( f \) is concave up on \((-\infty, -\frac{2}{3}]\).
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Corollary 2 implies that \( f \) is concave up on \( (-\infty, -\frac{2}{3}] \) and concave down on \([ -\frac{2}{3}, 0] \).
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Corollary 2 implies that \( f \) is concave up on \( (-\infty, -\frac{2}{3}] \) and concave down on \([-\frac{2}{3}, 0]\) and concave up on \([0, +\infty)\). Because of the two concavity changes,
\[ f(x) = 3x^4 + 4x^3 \]

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Corollary 2 implies that \( f \) is concave up on \((-\infty, -\frac{2}{3}]\) and concave down on \([-\frac{2}{3}, 0]\) and concave up on \([0, +\infty)\). Because of the two concavity changes, \( f \) has inflection points at \(-\frac{2}{3}\) and at 0.
\[ f(x) = 3x^4 + 4x^3 \]

The graph follows. The minimum is indicated in blue and the inflection points in green.
Problem. \( f(x) = x + x^{2/3} \). Find where \( f \) is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch \( f \) showing all these things.
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\( x^{2/3} \) is continuous on \((-\infty, +\infty)\).
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\( x^{-4/3} = (x^{-2/3})^2 = \left(\frac{1}{x^{2/3}}\right)^2 \) which is positive for all \( x \neq 0 \).
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\[ x^{-4/3} = (x^{-2/3})^2 = \left(\frac{1}{x^{2/3}}\right)^2 \text{ which is positive for all } x \neq 0. \]

So \( f''(x) < 0 \) for all \( x \neq 0 \).
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$x^{-4/3} = (x^{-2/3})^2 = \left(\frac{1}{x^{2/3}}\right)^2$ which is positive for all $x \neq 0$. So $f''(x) < 0$ for all $x \neq 0$. Now by Corollary 2, $f$ is concave down on $(-\infty, 0)$ and on $(0, +\infty)$. 

Problem. \( f(x) = x + x^{2/3} \). Find where \( f \) is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch \( f \) showing all these things.

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\[ f(x) = x + x^{2/3} \]

\[
f'(x) = 1 + \frac{2}{3}x^{-1/3} = \frac{x^{1/3} + \frac{2}{3}}{x^{1/3}}
\]

\[
= \frac{x^{1/3} + \frac{2}{3}}{x^{1/3}} \cdot \frac{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}}{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}} \cdot \frac{x^{2/3}}{x^{2/3}}
\]

\[
= \frac{x + \frac{8}{27}}{x} \cdot \frac{x^{2/3}}{(x^{1/3})^2 - \frac{2}{3}x + \frac{4}{9}}.
\]
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\[
(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9} = (x^{1/3})^2 - 2\frac{1}{3}x^{1/3} + \frac{1}{9} + \frac{1}{3} = (x^{1/3} - \frac{1}{3})^2 + \frac{1}{3} \geq \frac{1}{3} > 0 \text{ for all } x.
\]
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\]

\( (x^{1/3} - \frac{1}{3})^2 + \frac{1}{3} \geq \frac{1}{3} > 0 \) for all \( x \). Also, \( x^{2/3} = (x^{1/3})^2 > 0 \) for all \( x \neq 0 \).
\( f(x) = x + x^{2/3} \)

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So

\[
\frac{x^{2/3}}{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}} > 0 \quad \text{for all } x \neq 0.
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\[
(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9} = (x^{1/3})^2 - 2 \frac{1}{3}x^{1/3} + \frac{1}{3} + \frac{1}{3} = (x^{1/3} - \frac{1}{3})^2 + \frac{1}{3} \geq \frac{1}{3} > 0 \text{ for all } x. \text{ Also, } x^{2/3} = (x^{1/3})^2 > 0 \text{ for all } x \neq 0.
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If \( x < -\frac{8}{27} \), then \( x < x + \frac{8}{27} < 0 \).
\[ f(x) = x + x^{2/3} \]

\[
\begin{align*}
  f'(x) &= 1 + \frac{2}{3}x^{-1/3} = \frac{x^{1/3} + \frac{2}{3}}{x^{1/3}} \\
  &= \frac{x^{1/3} + \frac{2}{3}}{x^{1/3}} \cdot \frac{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}}{\frac{(x^{1/3})^2}{x^{2/3}} - \frac{2}{3}x + \frac{4}{9}} \cdot \frac{x^{2/3}}{x^{2/3}} \\
  &= \frac{x + \frac{8}{27}}{x} \cdot \frac{x^{2/3}}{(x^{1/3})^2 - \frac{2}{3}x + \frac{4}{9}}.
\end{align*}
\]

\[
(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9} = (x^{1/3})^2 - 2\frac{1}{3}x^{1/3} + \frac{1}{9} + \frac{1}{3} = (x^{1/3} - \frac{1}{3})^2 + \frac{1}{3} \geq \frac{1}{3} > 0 \text{ for all } x. \text{ Also, } x^{2/3} = (x^{1/3})^2 > 0 \text{ for all } x \neq 0.
\]

So

\[
\frac{x^{2/3}}{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}} > 0 \text{ for all } x \neq 0.
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If \( x < -\frac{8}{27} \), then \( x < x + \frac{8}{27} < 0 \). So \( \frac{x + \frac{8}{27}}{x} > 0 \),
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\[ = \frac{x^{1/3} + \frac{2}{3}}{x^{1/3}} \cdot \frac{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}}{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}} \cdot \frac{x^{2/3}}{x^{2/3}} \]

\[ = \frac{x + \frac{8}{27}}{x} \cdot \frac{x^{2/3}}{(x^{1/3})^2 - \frac{2}{3}x + \frac{4}{9}}. \]

\[ (x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9} = (x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{1}{9} + \frac{1}{3} = (x^{1/3} - \frac{1}{3})^2 + \frac{1}{3} \geq \frac{1}{3} > 0 \text{ for all } x. \text{ Also, } x^{2/3} = (x^{1/3})^2 > 0 \text{ for all } x \neq 0. \]

So

\[ \frac{x^{2/3}}{(x^{1/3})^2 - \frac{2}{3}x^{1/3} + \frac{4}{9}} > 0 \text{ for all } x \neq 0. \]

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If \( x < -\frac{8}{27} \), then \( x < x + \frac{8}{27} < 0 \). So \( \frac{x + \frac{8}{27}}{x} > 0 \), and thus \( f'(x) > 0 \); i.e., \( f'(x) > 0 \) on \( (-\infty, -\frac{8}{27}) \).
\[ f(x) = x + x^{2/3} \]

Since \( f'(x) > 0 \) on \( (-\infty, -\frac{8}{27}) \), by Theorem 1 we have that \( f \) is increasing on \( (-\infty, -\frac{8}{27}] \).
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Since $f'(x) > 0$ on $(-\infty, -\frac{8}{27})$, by Theorem 1 we have that $f$ is increasing on $(-\infty, -\frac{8}{27}]$.

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Because of this information about where \( f \) is increasing and decreasing, we have that \( f(-\frac{8}{27}) \) is a local max and \( f(0) \) is a local min.
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\[
f(-\frac{8}{27}) = -\frac{8}{27} + \left(-\frac{8}{27}\right)^{2/3} = -\frac{8}{27} + \frac{4}{9} = \frac{4}{27}, \text{ and } f(0) = 0.
\]
$f(x) = x + x^{2/3}$

The graph follows. The local extrema are indicated in blue.
Problem. $f(x) = (x^2 - 1)^2$. Find where $f$ is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch $f$ showing all these things.
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\( f(x) \) is a polynomial and so continuous everywhere.
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\( f'(x) = 2(x^2 - 1)2x = 4x(x + 1)(x - 1) \).
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From what we did earlier, we learned that $f$ is decreasing on $(-\infty, -1]$ and increasing on $[-1, 0]$ and decreasing on $[0, 1]$ and increasing on $[1, +\infty)$. 
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Since \( f \) is decreasing on \((-\infty, -1]\) and increasing on \([-1, 0]\), so \( f(-1) \) is a local minimum value.
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\[ f''(x) = 4[3x^2 - 1] = 12 \left(x^2 - \frac{1}{3}\right) = 12 \left(x + \frac{1}{\sqrt{3}}\right) \left(x - \frac{1}{\sqrt{3}}\right). \]
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If \( x < -\frac{1}{\sqrt{3}} \), then \( (x - \frac{1}{\sqrt{3}}) < (x + \frac{1}{\sqrt{3}}) < 0 \), and so \( f''(x) = 12 \left( x - \frac{1}{\sqrt{3}} \right) \left( x + \frac{1}{\sqrt{3}} \right) > 0 \); if \( -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \), then \( (x - \frac{1}{\sqrt{3}}) < 0 < (x + \frac{1}{\sqrt{3}}) \), and so \( f''(x) = 12 \left( x - \frac{1}{\sqrt{3}} \right) \left( x + \frac{1}{\sqrt{3}} \right) < 0 \);

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The function $f$ is concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right]$ and concave down on $\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$ and concave up on $\left[\frac{1}{\sqrt{3}}, \infty\right)$. Because of these changes in concavity, $f$ has inflection points at $\pm \frac{1}{\sqrt{3}}$. 

$f(x) = (x^2 - 1)^2$
$f(x) = (x^2 - 1)^2$

The graph follows. The local extrema are indicated in blue and the inflection points in green.
Problem. \( f(x) = x^4(x - 2)^4 \). Find where \( f \) is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Carefully justify each claim. Sketch \( f \) showing all these things.
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\[
f'(x) = 4x^3(x - 2)^4 + x^4 4(x - 2)^3 = 4x^3(x - 2)^3[x - 2 + x] = 4x^3(x - 2)^3[2x - 2] = 8x^3(x - 2)^3(x - 1).
\]
\[ f(x) = x^4(x - 2)^4 \]

If \( x < 0 \), then \( x - 2 < x - 1 < x < 0 \), and so
\[ f'(x) = 8(x - 2)^3(x - 1)x^3 < 0; \quad f'(x) < 0 \text{ on } (-\infty, 0). \]
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Now by Theorem 1, we see that \( f \) is decreasing on \((-\infty, 0]\),
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If $1 < x < 2$, then $x - 2 < 0 < x - 1 < x$, and so 
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If $2 < x$, then $0 < x - 2 < x - 1 < x$, and so 
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So \( 0 = f(0) \) is a local minimum value, and \( 1 = f(1) \) is a local maximum value, and \( 0 = f(2) \) is a local minimum value.
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Now we calculate and factor the second derivative.
\[ f(x) = x^4(x - 2)^4 \]

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\[ f''(x) = 24x^2(x - 2)^3(x - 1) + 24x^3(x - 2)^2(x - 1) + 8x^3(x - 2)^3 \]
\[ = 8x^2(x - 2)^2[3(x - 2)(x - 1) + 3x(x - 1) + x(x - 2)] \]
\[ = 8x^2(x - 2)^2[3(x^2 - 3x + 2) + 3(x^2 - x) + (x^2 - 2x)] \]
\[ = 8x^2(x - 2)^2[7x^2 - 14x + 6] = 8x^2(x - 2)^2[7x^2 - 14x + 7 - 1] \]
\[ = 56x^2(x - 2)^2 \left[ (x^2 - 2x + 1) - \frac{1}{7} \right] = 56x^2(x - 2)^2 \left[ (x - 1)^2 - \frac{1}{7} \right] \]
\[ = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right). \]
{\textit{f}}(x) = x^4(x - 2)^4

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\[ 56x^2(x - 2)^2 > 0 \text{ as long as } 0 \neq x \neq 2. \]
\( f(x) = x^4(x - 2)^4 \)

\[
f'''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right).
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\[ 56x^2(x - 2)^2 > 0 \] as long as \( 0 \neq x \neq 2. \)

If \( x < 1 - \frac{1}{\sqrt{7}} \), then \( x - 1 - \frac{1}{\sqrt{7}} < x - 1 + \frac{1}{\sqrt{7}} < 0. \) So
\[ f(x) = x^4(x - 2)^4 \]

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If x < 1 - \( \frac{1}{\sqrt{7}} \), then x - 1 - \( \frac{1}{\sqrt{7}} \) < x - 1 + \( \frac{1}{\sqrt{7}} \) < 0. So

\[ f'''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right) > 0 \] as long as x \( \neq \) 0.
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as long as \( x \neq 0 \). So \( f''(x) > 0 \) on \(( -\infty, 0)\) and on \( \left( 0, 1 - \frac{1}{\sqrt{7}} \right) \).
\[ f(x) = x^4(x - 2)^4 \]

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56\(x^2(x - 2)^2 > 0\) as long as \(0 \neq x \neq 2\).

If \(x < 1 - \frac{1}{\sqrt{7}}\), then \(x - 1 - \frac{1}{\sqrt{7}} < x - 1 + \frac{1}{\sqrt{7}} < 0\). So \(f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right) > 0\) as long as \(x \neq 0\). So \(f''(x) > 0\) on \((-\infty, 0)\) and on \((0, 1 - \frac{1}{\sqrt{7}})\). So by Corollary 2, we see that \(f\) is concave up on \((-\infty, 0]\) and on \([0, 1 - \frac{1}{\sqrt{7}}]\). Because of the overlap, \(f\) is concave up on \((-\infty, 1 - \frac{1}{\sqrt{7}}]\).
\( f(x) = x^4(x - 2)^4 \)

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\[ f(x) = x^4(x - 2)^4 \]

\[ f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right). \]

56x^2(x - 2)^2 > 0 as long as 0 \(\neq\) x \(\neq\) 2.

If \(1 - \frac{1}{\sqrt{7}} < x < 1 + \frac{1}{\sqrt{7}}\), then 0 \(\neq\) x \(\neq\) 2 and

\[ x - 1 - \frac{1}{\sqrt{7}} < 0 < x - 1 + \frac{1}{\sqrt{7}}. \]

So
\[ f(x) = x^4(x - 2)^4 \]

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\[ f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right) < 0. \]
\[ f(x) = x^4(x - 2)^4 \]

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56x^2(x - 2)^2 > 0 as long as 0 \neq x \neq 2.

If 1 - \frac{1}{\sqrt{7}} < x < 1 + \frac{1}{\sqrt{7}}, then 0 \neq x \neq 2 and
\[ x - 1 - \frac{1}{\sqrt{7}} < 0 < x - 1 + \frac{1}{\sqrt{7}}. \] So
\[ f''(x) = 56x^2(x - 2)^2 \left(x - 1 - \frac{1}{\sqrt{7}}\right) \left(x - 1 + \frac{1}{\sqrt{7}}\right) < 0. \] So by
Corollary 2, we see that \( f \) is concave down on \([1 - \frac{1}{\sqrt{7}}, 1 + \frac{1}{\sqrt{7}}]\).
\[ f(x) = x^4(x - 2)^4 \]

\[ f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right) . \]

\[ 56x^2(x - 2)^2 > 0 \text{ as long as } 0 \neq x \neq 2. \]
\[ f(x) = x^4(x - 2)^4 \]

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56x^2(x - 2)^2 > 0 as long as 0 \(\neq\) x \(\neq\) 2.
If \(1 + \frac{1}{\sqrt{7}} < x\) and \(x \neq 2\), then \(0 < x - 1 - \frac{1}{\sqrt{7}} < x - 1 + \frac{1}{\sqrt{7}}\). So
\[ f(x) = x^4(x - 2)^4 \]

\[ f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right). \]

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\[ f(x) = x^4(x - 2)^4 \]

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56x^2(x - 2)^2 > 0 as long as 0 \( \neq \) x \( \neq \) 2.

If 1 + \( \frac{1}{\sqrt{7}} \) < x and x \( \neq \) 2, then 0 < x - 1 - \( \frac{1}{\sqrt{7}} \) < x - 1 + \( \frac{1}{\sqrt{7}} \). So

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So by Corollary 2, we see that \( f \) is concave up on \([1 + \frac{1}{\sqrt{7}}, 2]\) and on \([2, +\infty)\).
\[ f(x) = x^4(x - 2)^4 \]

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If \( 1 + \frac{1}{\sqrt{7}} < x \) and \( x \neq 2 \), then \( 0 < x - 1 - \frac{1}{\sqrt{7}} < x - 1 + \frac{1}{\sqrt{7}} \). So \( f''(x) = 56x^2(x - 2)^2 \left( x - 1 - \frac{1}{\sqrt{7}} \right) \left( x - 1 + \frac{1}{\sqrt{7}} \right) > 0 \). So by Corollary 2, we see that \( f \) is concave up on \( [1 + \frac{1}{\sqrt{7}}, 2] \) and on \( [2, +\infty) \). Because of the overlap, \( f \) is concave up on \( [1 + \frac{1}{\sqrt{7}}, \infty) \).
\[ f(x) = x^4(x - 2)^4 \]

Summarizing, \( f \) is concave up on \( (-\infty, 1 - \frac{1}{\sqrt{7}}] \)
\[ f(x) = x^4(x - 2)^4 \]

Summarizing, \( f \) is concave up on \(( -\infty, 1 - \frac{1}{\sqrt{7}} \)] and concave down on \([1 - \frac{1}{\sqrt{7}}, 1 + \frac{1}{\sqrt{7}}])

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Summarizing, $f$ is concave up on $\left(-\infty, 1 - \frac{1}{\sqrt{7}}\right]$ and concave down on $\left[1 - \frac{1}{\sqrt{7}}, 1 + \frac{1}{\sqrt{7}}\right]$ and concave up on $\left[1 + \frac{1}{\sqrt{7}}, \infty\right)$.

This means that $f$ has inflection points at $1 \pm \frac{1}{\sqrt{7}}$. $f$ does not have inflection points at 0 or 2 even though $f''(0) = 0 = f''(2)$. This is because the concavity does not change up to down or down to up at either 0 or 2. Remember this example, and do not ever assert that $a$ is an inflection point because $f''(a) = 0$!
$f(x) = x^4(x - 2)^4$

Summarizing, $f$ is concave up on $(-\infty, 1 - \frac{1}{\sqrt{7}}]$ and concave down on $[1 - \frac{1}{\sqrt{7}}, 1 + \frac{1}{\sqrt{7}}]$ and concave up on $[1 + \frac{1}{\sqrt{7}}, \infty)$. This means that $f$ has inflection points at $1 \pm \frac{1}{\sqrt{7}}$. 

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Remember this example, and do not ever assert that $a$ is an inflection point because $f''(a) = 0$!
\[ f(x) = x^4(x - 2)^4 \]

The graph follows.

The local extrema are indicated in blue and the inflection points in green.
Problem. \( f(x) = \frac{2x^2 + 1}{x^2 - 1} \). Find where \( f \) is increasing, decreasing, concave up and concave down. Find all local extrema and inflection points. Find any horizontal and vertical asymptotes. Carefully justify each claim. Sketch \( f \) showing all these things.
\[ f(x) = \frac{(2x^2 + 1)}{(x^2 - 1)} \]

\[
\begin{align*}
f(x) &= \frac{2x^2 + 1}{x^2 - 1} = \frac{2x^2 - 2 + 3}{x^2 - 1} = 2 + \frac{3}{x^2 - 1} \\
f'(x) &= \frac{-3}{(x^2 - 1)^2} 2x = 6 \frac{-x}{(x^2 - 1)^2} \\
f''(x) &= 6 \frac{-1(x^2 - 1)^2 - (-x)2(x^2 - 1)2x}{(x^2 - 1)^4} = 6 \frac{-x^2 + 1 + 4x^2}{(x^2 - 1)^3} \\
&= 6 \frac{3x^2 + 1}{(x^2 - 1)^3}
\end{align*}
\]

Theorem 1 implies \( f \) is increasing on \((\infty, 0)\) and \((-1, 0)\).
\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} \]

\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} = \frac{2x^2 - 2 + 3}{x^2 - 1} = 2 + \frac{3}{x^2 - 1} \]

\[ f'(x) = \frac{-3}{(x^2 - 1)^2} \cdot 2x = 6 \frac{-x}{(x^2 - 1)^2} \]

\[ f''(x) = 6 \frac{-1(x^2 - 1)^2 - (-x)2(x^2 - 1)2x}{(x^2 - 1)^4} = 6 \frac{-x^2 + 1 + 4x^2}{(x^2 - 1)^3} \]

\[ = 6 \frac{3x^2 + 1}{(x^2 - 1)^3} \]

\[ f(x) \text{ is not defined at } \pm 1. \]
\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} \]

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\[ f(x) \] is not defined at \( \pm 1 \). \( f \) is a rational function.
\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} = \frac{2x^2 - 2 + 3}{x^2 - 1} = 2 + \frac{3}{x^2 - 1} \]

\[ f'(x) = \frac{-3}{(x^2 - 1)^2} 2x = 6 \frac{-x}{(x^2 - 1)^2} \]

\[ f''(x) = 6 \frac{-1(x^2 - 1)^2 - (-x)2(x^2 - 1)2x}{(x^2 - 1)^4} = 6 \frac{-x^2 + 1 + 4x^2}{(x^2 - 1)^3} \]

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\[ f(x) \text{ is not defined at } \pm 1. \ f \text{ is a rational function. So } f \text{ is continuous wherever it is defined.} \]
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\( f(x) \) is not defined at \( \pm 1 \). \( f \) is a rational function. So \( f \) is continuous wherever it is defined. So \( f \) is continuous on \(( -\infty, -1)\) and \((-1, 1)\) and on \((1, +\infty)\).
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\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} \]

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\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} \]

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\((x^2 - 1)^2 > 0\) as long as \( x \neq \pm 1 \). So if \(-1 \neq x < 0\), then \(-x > 0\) and \( f'(x) > 0 \);
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\( (x^2 - 1)^2 > 0 \) as long as \( x \neq \pm 1 \). So if \( -1 \neq x < 0 \), then \( -x > 0 \) and \( f'(x) > 0 \); i.e., \( f'(x) > 0 \) on \((-\infty, -1)\) and on \((-1, 0)\).
\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} \]

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\[
f(x) = \frac{(2x^2 + 1)}{(x^2 - 1)}
\]

We have that \( f \) is increasing on \(( -\infty, -1)\) and on \((-1, 0]\).
\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} \]

We have that \( f \) is increasing on \((−∞, −1)\) and on \((-1, 0]\) (but not on \((−∞, −1) \cup (-1, 0].\))
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\[
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\((x^2 - 1)^2 > 0\) as long as \(x \neq \pm 1\). So if \(1 \neq x > 0\), then \(-x < 0\) and \(f'(x) < 0\);
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\((x^2 - 1)^2 > 0\) as long as \(x \neq \pm 1\). So if \(1 \neq x > 0\), then \(-x < 0\) and \(f'(x) < 0\); \(f'(x) < 0\) on \((0, 1)\) and on \((1, +\infty)\). Theorem 1 implies \(f\) is decreasing on \([0, 1)\) and on \((1, +\infty)\).
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$$f'(x) = 6 \frac{-x}{(x^2 - 1)^2}$$

$(x^2 - 1)^2 > 0$ as long as $x \neq \pm 1$. So if $1 \neq x > 0$, then $-x < 0$ and $f'(x) < 0$; $f'(x) < 0$ on $(0, 1)$ and on $(1, +\infty)$. Theorem 1 implies $f$ is decreasing on $[0, 1)$ and on $(1, +\infty)$ (but not on $[0, 1) \cup (1, +\infty)$.)

Because $f$ is increasing on $(-1, 0]$ and decreasing on $[0, 1)$, we see that $f(0) = -1$ is a local maximum value. There are no local minima.
\[ f(x) = \frac{2x^2 + 1}{x^2 - 1} \]

\[ f''(x) = 6 \frac{3x^2 + 1}{(x^2 - 1)^3} \]

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Corollary 2 implies \( f \) is concave up on \((-\infty, -1)\)

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Corollary 2 implies \(f\) is concave up on \((-\infty, -1)\) and concave down on \((-1, 1)\).
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Corollary 2 implies \( f \) is concave up on \( (-\infty, -1) \) and concave down on \( (-1, 1) \) and concave up on \( (1, +\infty) \). There are no inflection points.
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So \( y = 2 \) is a horizontal asymptote.

\[
\lim_{x \to \pm 1} x^2 - 1 = 0 \neq 3 = \lim_{x \to \pm 1} 2x^2 + 1,
\]

and so \( \lim_{x \to \pm 1} \left| \frac{2x^2 + 1}{x^2 - 1} \right| = \infty \).

So \( x = -1 \) and \( x = 1 \) are vertical asymptotes.
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\[ 2x^2 + 1 \geq 1 > 0 \] for all \( x \). Above we showed that \( x^2 - 1 > 0 \) on \( (-\infty, -1) \) and on \( (1, +\infty) \) while \( x^2 - 1 < 0 \) on \( (-1, 1) \).
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$$\lim_{x \to -1^-} \frac{2x^2 + 1}{x^2 - 1} = \infty = \lim_{x \to 1^+} \frac{2x^2 + 1}{x^2 - 1},$$

and

$$\lim_{x \to -1^+} \frac{2x^2 + 1}{x^2 - 1} = -\infty = \lim_{x \to 1^-} \frac{2x^2 + 1}{x^2 - 1}.$$
$$f(x) = \frac{(2x^2 + 1)}{(x^2 - 1)}$$

The graph follows in red with the asymptotes in green.