Module 3 - Linear versus Exponential

Department of Mathematics

University of Hawaiʻi at Mānoa

Math 100

Two Important Change Patterns

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Another way to say the same thing: for linear grow (decay), you **add (subsctract)** the same amount per unit of time, and for exponential growth(decay) you **multiply(divide)** by the same amount for each unit of time.

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Example 2:
You save money by putting \$10 under your mattress each week. The amount of money you have grows **linearly**: each week you **add** \$10 to the amount of money you have.

Examples of Exponential Growth

Example 1:
A disease is spread from one individual to three other people every day. The amount of cases grows **exponentially**: every day, you **multiply** the amount by 3.

Example 2:
A colony of bacteria doubles in size every hour. This is **exponential** growth: every hour, you **multiply** the number of bacteria by two.

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During ten weeks the price will increase by $4 \times 10 = 40 cents$,

and will become $4.00 + 40 = \$4.40$

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Where Q_0 is the initial amount and, m is the rate of change, and t is the time that has gone by.

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This is the equation of a line, with slope m.
Another way to commonly write this is

$$y = b + m\,x$$

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Now the pattern becomes clear: n years from now the car is worth

$$12000 \times (1 - 0.1)^n = 12000 \times 0.9^n$$

For example, 20 years from now, the car is worth $12000 \times 0.9^{20} \approx \$1,459$

Exponential Decay

Notice that this formula for the worth after n years

$$12000 \times (1 - 0.1)^n = 12000 \times 0.9^n$$

is the same as the first formula

$$Q = Q_0(1 + r)^n$$

except here our $r = -0.1$ or -10% since we have **exponential decay**.

Exponential Models

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- Q is the quantity being discussed, and t is the time passed. Recall that these are called **variables**, with Q being the **dependent** variable, and t the **independent** variable.
- Q_0 is the initial quantity, and r is the rate of increase / decrease. These are called **constants**: they are usually fixed as part of the problem.

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- Types 1 and 3 use the whole equation $Q = Q_0 \times (1+r)^t$; types 2 and 4 really just use the 'multiplier' part $(1+r)^t$.
- Types 3 and 4 (almost always) need you to use logs, but 1 and 2 are algebraically simpler and do not.

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- Doubling time is an intuitive measure of how fast something is growing.

- It also gives a good way of predicting the quantity of something in the future.

COVID-19 Exponential Spread

An early estimate of doubling time for COVID-19 in a collection of countries was found to be 5 days. By what factor does it grow in 10 days? In 15 days?

1. If the disease starts in 1 person, it becomes 2 people after 5 days.

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1. If the disease starts in 1 person, it becomes 2 people after 5 days. Then, it becomes 4 people after another 5 days.Thus 1 person has turned into 4 people during 10 days, and the factor is 4.
2. Similarly, during 15 days $1 \to 2 \to 4 \to 8$, and the factor is 8.

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- After 3 more years it should double again...
- After n years: $10 \times (1.25)^n$, so after 6 years we have $10 \times (1.25)^6 = 38.14$ which is about 38 people. The actual doubling time is a bit more that 3 years.

Finding The Doubling Time

In many cases, exponential growth is specified simply by giving a rate of growth in percentage points.

Suppose the initial value of our quantity is Q_0, and it grows by P% each unit of time. Let $r = \frac{P}{100}$ be the fractional growth rate. Then we know that after time t the quantity will have the value

$$Q_t = Q_0 \times (1 + r)^t$$

We can find the doubling time from the above equation. But to better understand how we do it, let's review logarithms.

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- $\log xy = \log x + \log y$.
- $\log \frac{x}{y} = \log x - \log y$.
- $\log x^y = y \log x$.
- It may be useful to have a *default* base in mind. Say, you can interpret $\log x$ as $\log_{10} x$. Just make sure you're consistent. Most calculators (and google) use base 10 as default.

Back to Finding Doubling Time

We know that after time t the quantity with initial value Q_0 and a fractional growth rate r will have the value

$$Q_t = Q_0 \times (1+r)^t$$

If $t = T_{double}$, then we have:

$$2Q_0 = Q_0 \times (1+r)^{T_{double}}, \qquad \text{so } 2 = (1+r)^{T_{double}}$$

Taking logarithm of both sides we get:

$$\log 2 = \log(1+r)^{T_{double}} = T_{double} \log(1+r)$$

So, we find:

$$T_{double} = \frac{\log 2}{\log(1+r)}$$

Doubling Time For Company Again

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Due to the properties of logarithms, it doesn't matter in the formula if you use log_{10} or log with some other base. However, you must use **the same** base in the numerator and denominator.

Approximate The Doubling Time

We know the exact formula to find the doubling time: $T_{double} = \frac{\log 2}{\log(1+r)}$, where r is the fractional growth rate.

- Sometimes, using logarithms in our computations can be a bit tedious. It turns out that if r is small (which often is the case), then $\log_{10}(1+r) \approx 0.434r$.

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- Using the base 10 the exact formula becomes: $T_{double} = \frac{\log_{10} 2}{\log_{10}(1+r)}$
- Plugging in our approximation for $\log_{10}(1+r)$ we get:

$$T_{double} \approx \frac{\log_{10} 2}{0.434r} \approx \frac{0.7}{r} = \frac{0.7}{P/100} = \frac{70}{P}$$

where P is the growth rate in percents.

Oil Consumption Increase

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Solution. By the approximate doubling time formula with P=2.2%,

$$T_{double} \approx \frac{70}{P} = \frac{70}{2.2} \approx 32 \text{ years}$$

The factor is

$$2^{t/T_{double}} \approx 2^{10/32} \approx 1.24$$

Exponential Decay and Half-life

With exponential decay, the quantity decreases and after some time becomes a half of what it was initially. The time required for each halving is called the *half-life*.

- If T_{half} denotes the half-life, then we can show that after time t, the exponentially decreasing quantity has the value:

$$\text{new value} = \text{initial value} \times \left(\tfrac{1}{2}\right)^{t/T_{half}}$$

Notice that $\left(\frac{1}{2}\right)^{t/T_{half}} = 2^{-t/T_{half}}$

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- Which is similar to

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- Half of the drug remains after 13 days; half of this half, which is a quarter, remains after 26 days, half of his quarter, which is $\frac{1}{8}$, remains after 39 days; another 39 hours will leave an eighth of this $\frac{1}{8}$. Continuing like this, we have $\frac{1}{64}$ after 78 days.
- Another 78 days will leave the bloodstream with $\frac{1}{64}$ of this $\frac{1}{64}$. Thus after 78 days there is $\frac{1}{64} \times \frac{1}{64} = \frac{1}{4096}$ of the initial quantity.

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- We can use the formula with $T_{half} = 13$:

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  $$(\frac{1}{2})^{t/T_{half}} = (\frac{1}{2})^{156/13} = (\frac{1}{2})^{12} = \frac{1}{4096}.$$

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- Note that P here is in percents per unit of time. One makes use of this formula in a way similar to that of doubling time formula.