Constructing a weak subset of a random set
Joint work with Lu Liu

Bjørn Kjos-Hanssen

Department of Mathematics
University of Hawai‘i at Mānoa
USA

`bjoern.kjos-hanssen@hawaii.edu`

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Reverse mathematics

- Ramsey's Theorem for pairs
  - each 2-coloring of $[\omega]^2$ has an infinite homogeneous set
  - for each Δ^0_2 definition of a possible set A, an infinite subset or co-subset of A exists
- WKL (Weak König's Lemma: each infinite finitely branching tree has an infinite path
- WWKL (each tree with many branches at each level has a path)
- DNR (there is a function $f : \omega \to \omega$ with $f(e) \neq \varphi_e(e)$)

Results

- $\mathsf{RT}^2_2 \nvdash \mathsf{WKL}_0$ (Liu 2012)
- $\mathsf{RT}^2_2 \nvdash \mathsf{WWKL}_0$ (Liu 2015)
- $\mathsf{DNR} \nvdash \mathsf{WWKL}_0$ (Ambos-Spies, K., Lempp, Slaman 2004)

Corresponding computability results:

- Each Δ^0_2 set has an infinite subset or co-subset that computes no completion of Peano Arithmetic (Liu 2012)
- Each Δ^0_2 set has an infinite subset or co-subset that computes no ML-random set (Liu 2015)
- There is a DNR function that computes no ML-random set (Ambos-Spies, K., Lempp, Slaman 2004)
- DNR functions are Muchnik equivalent to infinite subsets of random sets (mentioned in Greenberg, J. Miller 2009)

General question

If A has some computational power and B is similar to A, does B retain that power?

General question

If A has some computational power and $B \subseteq A$, B infinite, does B retain that power?

Two examples of sets where you *can* recover nontrivial information from any subset:

- Very sparse sets
- Sets of prefixes $\{\sigma : \sigma \text{ prefix of } A\}$

My question

AIM Workshop, Palo Alto, 2006

Does every Martin-Löf random set have an infinite subset that computes no ML-random set?

Answer: No.

Theorem (K., 2009)

Some ML-random has an infinite subset that computes no ML-random set.

It raised another question: is this typical?

Theorem (K., 2011)

Almost every ML-random has an infinite subset that computes no ML-random set.

Theorem (K. and Liu, 2019)

Every ML-random has an infinite subset that computes no ML-random set.

In this form a proof sketch (personal communication) was given by J. Miller, 2011. But we have strengthenings.

Theorem (K. and Liu, 2019)

Every ML-random has an infinite subset that computes no real of positive effective Hausdorff dimension.

Theorem (K. and Liu, 2019)

Every effectively incompressible set has an infinite subset that computes no real of positive effective Hausdorff dimension.

But effectively incompressible = CB-random studied by Brodhead, Downey, Ng. So

Theorem (K. and Liu, 2019)

Every CB-random has an infinite subset that computes no real of positive effective Hausdorff dimension.

Also:

Theorem (K. and Liu, 2019)

Every CB-random has an infinite subset that computes no real of positive complex packing dimension.

Liu's approach

Definition (Beigel et al.)

Given a set $S \subseteq j^{<\omega}$, an ℓ-*enumeration* of S is a function $g : \omega \to \mathrm{Fin}(j^{<\omega})$ such that $|g(n)| \leq \ell$ and $g(n) \cap S \cap j^n \neq \emptyset$ for all n. A *bounded enumeration* of S is an ℓ-enumeration for some $\ell \in \omega$.

Theorem

Given a set A that is not effectively compressible, let $S^u \subseteq j^{<\omega}$, $u \in \omega$, be a family of co-c.e. sets such that none of the $S^u, u \in \omega$ admits computable bounded enumeration. Then there exists an infinite set $G \subseteq A$ such that none of the S^u admits any bounded enumeration computable in G.

Corollary

For any 1-random set A, there exists an infinite subset G of A such that G does not compute any set with positive effective Hausdorff dimension.

The corollary follows by noting:

- the sequence of trees defining "positive effective Hausdorff dimension" does not admit a computable bounded enumeration,
- 1-randomness implies not being effectively compressible, and
- the infinite subset G of A gives the bounded enumeration (in fact, 1-enumeration) given by $g(n) = \{G \upharpoonright n\}$.

The trees can be taken to be

$$T_{m,c} = \{\sigma \in 2^{<\omega} : (\forall k < |\sigma|)\, K(\sigma \restriction k) \geq k/m - c\}$$

as X has positive effective Hausdorff dimension if and only if

$$\exists m, c \quad \forall n \quad X \restriction n \in T_{m,c}.$$

If $T_{m,c}$ has a computable ℓ-enumeration then we can describe $\sigma \in T_{m,c}$ by giving $n = |\sigma|$ and the index of σ in the list of up to ℓ elements of $T_{m,c} \cap 2^n$. This would show $K(\sigma) \leq^+ 2(\log n + \log l)$ which for large n contradicts $K(\sigma) \geq |\sigma|/m - c$.

Complex packing dimension

Let the collection of all infinite computable subsets of ω be denoted by $\mathfrak{C}$. Let $p : \omega \to \omega$. Let d be the Hamming distance function. For $X, Y \in 2^\omega$ and $N \subseteq \omega$ we define a notion of proximity, or similarity, by

$$X \sim_{p,N} Y \quad \iff \quad (\exists n_0)(\forall n \in N,\ n \geq n_0)(d(X \upharpoonright n, Y \upharpoonright n) \leq p(n)).$$

Definition

The *effective Hausdorff dimension* of $A \in 2^\omega$ is

$$\dim_H(A) = \liminf_{n\in\omega} \frac{K(A \upharpoonright n)}{n}.$$

The *complex packing dimension* of $A \in 2^\omega$ is

$$\dim_{cp}(A) = \sup_{N\in\mathfrak{C}} \inf_{n\in N} \frac{K(A \upharpoonright n)}{n}.$$

The *effective packing dimension* of $A \in 2^\omega$ is

$$\dim_p(A) = \limsup_{n\in\omega} \frac{K(A \upharpoonright n)}{n}.$$

$$0 \leq \dim_H(A) \leq \dim_{cp}(A) \leq \dim_p(A) \leq 1.$$

The inequality $\dim_H(A) \leq \dim_{cp}(A)$ uses the fact that each cofinite set $N \subseteq \omega$ is in $\mathfrak{C}$. The inequality $\dim_{cp}(A) \leq \dim_p(A)$ uses the fact that each $N \in \mathfrak{C}$ is an infinite subset of ω.

Complex packing dimension

- Freer and Kjos-Hanssen (2013): Complex packing dimension not Medvedev above stochastically bi-immune.
- Let $p(n) = o(\frac{n}{\log n})$ and let $N \in \mathfrak{C}$. If $X \in \mathrm{MLR}$ and $X \sim_{p,N} Y$ then $\dim_{cp}(Y) = 1$.
- Contrast with *Dimension 1 sequences are close to randoms* by Greenberg, Miller, Shen, and Westrick (2018): Coarsely random same as effective Hausdorff dimension 1.
- Conjecture? "If $\dim_{cp}(Y) = 1$ then $\exists X \in \mathrm{MLR}$, $X \sim_{p,N} Y$".

New question

Does each real of positive *complex packing dimension* compute a real of positive *effective Hausdorff dimension*?

Corollary

For any 1-random set A, there exists an infinite subset of A, namely G, such that G does not compute any set with positive complex packing dimension.

Proof.

This time take the trees to be

$$T_{m,M,c} = \{\sigma \in 2^{<\omega} : (\forall k \in M)\, K(\sigma \upharpoonright k) \geq k/m - c\}$$

where $m > 0, c, m \in \omega, M \in \mathfrak{C}$, and $\mathfrak{C}$ is the collection of all infinite computable sets. These trees are all co-c.e. The fact that they are not *uniformly* co-.c.e. is not a problem since the main theorem uses a construction that deals with each tree in forcing requirements. □

Liu's construction

We say $X \in 2^\omega$ is a *k-partition* iff

- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1}$;
- $\bigcup_{i=0}^{k-1} X_i = \omega$.

A class $Q \subseteq 2^\omega$ is a *k-partition class* iff for every $X \in Q$, X is a k-partition.

Definition (Mathias condition)

A *Mathias condition* is a pair (σ, X) with $\sigma \in 2^{<\omega}$ and $X \in 2^{\omega}$.
We say that (τ, Y) *extends* the Mathias condition (σ, X) iff $\sigma \preceq \tau$ and $Y/\tau \subseteq X/\sigma$. Write $(\tau, Y) \leq (\sigma, X)$ to denote the extension relation.
We say that a set G *satisfies* the Mathias condition (σ, X) if $\sigma \prec G$ and $G \subseteq X/\sigma$.

Definition (Tree forcing conditions)

The forcing conditions we use to construct G are tuples

$$(k, \sigma_0, \ldots, \sigma_{k-1}, Q),$$

where $k > 0$, $\sigma_i \in 2^{<\omega}$,

- $\sigma_i \subseteq A$ for all $i \leq k-1$,
- Q is a nonempty Π^0_1 k-partition class,
- for every $X_0 \oplus \cdots \oplus X_{k-1} \in Q$ and every $i \leq k-1$, $\sigma_i \subseteq X_i \cap A$.

We regard each $X_0 \oplus \cdots \oplus X_{k-1} \in Q$ as representing k many Mathias conditions (σ_i, X_i), $i < k$.

Definition

We say that a condition $d' = (k', \sigma'_0, \dots, \sigma'_{k'-1}, Q')$ *extends* a condition $d = (k, \sigma_0, \dots, \sigma_{k-1}, Q)$, (henceforth $d' \le d$), if there is a function $f : k' \to k$ such that for all $i < k'$,

$$\forall Y_0 \oplus \cdots \oplus Y_{k'-1} \in Q' \quad \exists X_0 \oplus \cdots \oplus X_{k-1} \in Q \\ \left[(\sigma'_i, Y_i) \le (\sigma_{f(i)}, X_{f(i)})\right].$$

In this case, we say that

- f witnesses the extension $d' \le d$;
- part i of the condition d' refines part $f(i)$ of the condition d.

Definition

We say that a set G *satisfies* condition $(k, \sigma_0, \ldots, \sigma_{k-1}, Q)$ iff there is an $X_0 \oplus \cdots \oplus X_{k-1} \in Q$ such that G satisfies some (σ_i, X_i). In this case, we also say that G satisfies $(k, \sigma_0, \ldots, \sigma_{k-1}, Q)$ *on part* i.

We assume that for each Turing functional Ψ there exists ℓ_Ψ depending on Ψ such that for every X, Ψ^X is an ℓ_Ψ-enumeration with $\Psi^X(m)\downarrow \to \Psi^X(m) \subseteq j^m$.

For each Turing functional Ψ and $u \in \omega$, we need to satisfy the *requirement* $\mathcal{R}^u_\Psi$:

$$\Psi^G \text{ is not an } \ell_\Psi\text{-enumeration of } S^u \text{ if } \Psi^G \text{ is total.} \qquad (\mathcal{R}^u_\Psi)$$

Definition

We say condition d *forces* requirement $\mathcal{R}$ *on part* i iff every G satisfying d on part i also satisfies requirement $\mathcal{R}$. We say condition d *forces* requirement $\mathcal{R}$ iff it forces $\mathcal{R}$ on all parts.

Definition

We say part i of condition $c = (k, \sigma_0, \ldots, \sigma_{k-1}, Q)$ is *acceptable* iff there exists $X_0 \oplus \cdots \oplus X_{k-1} \in Q$ such that $X_i \cap A$ is infinite, where A is the given ML-random set.

A sample combinatorial idea used

Definition

A collection $V_1, \ldots, V_n$ of sets is *not k-dispersed* if it can be partitioned into k subcollections each having nonempty intersection.

How dispersedness is used

- The sets V_i are sets of strings that we may force Ψ^G into and that each may or may not have high effective Hausdorff dimension.
- We split into cases depending on how dispersed the sets V_i are and whether they contain complex strings.
- If they are not too dispersed and intersect S^u at length m then we can give a bounded enumeration of S^u at length m, by picking one element from each intersection.
- To find something that Gauss works on, divide his department up into a small number of research groups, and select a topic common to each group. One of those topics is of interest to Gauss! (...One of those strings is highly complex.)

Dispersedness

Invited speakers at this conference, last names:

BARTOSOVA
FISCHER
FORTNOW
HASKELL
IEMHOFF
MARQUIS
PATEL
SOSKOVA

Divide them into groups where each group has a shared letter.

$V_1 = \{\texttt{BARTOSOVA,HASKELL,MARQUIS,PATEL,SOSKOVA}\}$
$V_2 = \{\texttt{FISCHER,FORTNOW,IEMHOFF}\}$
So the set of names is not 2-dispersed. This is optimal: there is no letter shared by all.

$V_1 = \{\texttt{BARTOSOVA,HASKELL,MARQUIS,PATEL,SOSKOVA}\}$
$V_2 = \{\texttt{FISCHER,FORTNOW,IEMHOFF}\}$
So the set of names is not 2-dispersed. This is optimal: there is no letter shared by all.
Actually there are even two disjoint names.

CB-randomness and effective compressibility

Definition

- A is *not Martin-Löf random* if for each c, there is an n with $K(A \upharpoonright n) \leq n - c$.
- A is *effectively compressible* if there exists a computable function $f : \omega \to \omega$ such that $K_U(A \upharpoonright f(n)) \leq f(n) - n$.

CB-randomness

Brodhead, Downey and Ng (2012) defined *computably bounded finite randomness*.

Definition

- A Martin-Löf test $\{U_n\}_{n\in\omega}$ is *computably bounded* if there is some total computable function f such that the cardinality of the set of strings defining U_n is at most $f(n)$ for every n.
- A set if CB-random if it passes each computably bounded Martin-Löf test.

Each CB-random set has effective packing dimension 1.

Theorem

Effectively incompressible = CB-random

One direction:
If $K(A \upharpoonright f(n)) \leq f(n) - n$ then A belongs to

$$F_n := \{B : K(B \upharpoonright f(n)) \leq f(n) - n\}$$

which is a finite c.e. set of measure at most 2^{C-n}.

Other direction: use:

Definition

A set A is weakly CB-random if there is no computable f and uniformly c.e. sequence $F_n \subseteq \{0,1\}^{f(n)}$ such that $\mu F_n \leq 2^{-n}$ for each n and $A \in \cap_n [F_n]$.

Theorem

Weak CB-randomness is the same as CB-randomness.

Proof.

Suppose $\{V_e\}$ is a CB-test: $|V_e| \le f(e)$ and $\mu V_e \le 2^{-e}$ for each e. Now pick a length $n_e \ge e + \log_2 f(e)$. Let $R_e = \{\sigma \restriction n_e : \sigma \in V_e\}$.

$$\begin{aligned}
[R_e] &= [\{\sigma : \sigma \in V_e, |\sigma| \le n_e\}] \cup [\{\sigma \restriction n_e : \sigma \in V_e, |\sigma| > n_e\}] \\
\mu R_e &\le \mu V_e + f(e) 2^{-n_e} \\
&\le 2^{-e} + 2^{-e} = 2^{-(e-1)}
\end{aligned}$$

So let $U_e = R_{e+1}$ and the U_e's form a WCB-test with $\cap_e U_e \supseteq \cap_e V_e$. □

Effective incompressibility via Schnorr's theorem

TFAE

- For some ML-test U, we have $A \in \cap_{n\in\omega} U_n$
- For some ML-test U, we have $\forall n \quad \exists k \quad [A \upharpoonright k] \subseteq U_n$
- For some ML-test U, we have $\forall c \quad \exists n \quad [A \upharpoonright n] \subseteq U_c$
- $\forall c \quad \exists n \quad K(A \upharpoonright n) \leq n - c$

Now we "effectivize" this: TFAE

A is not CB-random For some computable f, for some ML-test U, we have $\forall c \quad [A \restriction f(c)] \in U_c$

A is effectively compressible For some computable f, $\forall c \quad K(A \restriction f(c)) \leq f(c) - c$

Proof.

Effectively compressible implies not CB-random since $\mu\{A : \exists n K(A \restriction n) \leq n - c\} \leq 2^{M-c}$ for a constant M. Effectively incompressible implies weakly CB-random: use a bounded request set (see e.g. Nies' book, Theorem 3.2.9). □

Where CB-randomness fits in

	comeager	meager
computable	Kurtz random	Schnorr random
c.e.	CB-random	ML-random

Blocks

There is an even weaker version, block-CB randomness, which suffices for our main theorem.

Definition

A block-CB test uses sets U_n that are of the form $U_n = \{X : X \upharpoonright I_n \in F_n\}$ where the I_n are computable disjoint intervals and the F_n are uniformly c.e. finite sets of cardinality at most $1/2 \cdot 2^{|I_n|}$ so that $\mu(U_n) \leq 1/2$.

Theorem

Block-CB random (for uniformly computable sets F) implies Kurtz random.

Proof.

Suppose A belongs to $[T]$ which has measure 0 for a computable tree T. Find a length n_0 giving measure at most $1/2$. How do we choose n_1 to get measure at most $1/2$ for the interval $T \cap [n_0, n_1]$? If a fraction $\geq \epsilon$ of the $2^{n_1 - n_0}$ strings in $[n_0, n_1]$ appear in T then the measure of $T \upharpoonright n_1$ is at least $\epsilon 2^{n_1 - n_0} 2^{-n_1} = \epsilon 2^{-n_0}$. So if we take $\epsilon = 1/2$, choose n_1 such that $\mu T \upharpoonright n_1 < 2^{-n_0 - 1}$. And iteratively choose n_{s+1} such that

$$\mu(T \upharpoonright n_{s+1}) < 2^{-n_s - 1}.$$

□

Same idea shows:

Theorem

Block-CB random implies CB random.

Proof.

We do this by taking large blocks: If A belong to F_n which has probability at most 2^{-n} and contained in $[0, m_n]$ then take G_n to be the interval $I = [m_n, m_{f(n)}]$ information in $\cap_n F_n$,

$$G_n = \{X : (\exists Y) X \upharpoonright I = Y \upharpoonright I, Y \in \cap_n F_n\}$$

where f is so large that $\mu(F_{f(n)}) < 2^{-m_n - 1}$, which we ensure by letting $f(n) = m_{n+1} (\geq m_n + 1)$ say. □

So the block-nature of the proof in our main theorem is not important.

Theorem (Brodhead, Downey, Ng)

A Turing incomplete c.e. real can be CB-random.

Question

Does every CB-random have an infinite subset that is *still CB-random* and computes no set of positive complex packing dimension?

Question

Does every effective-packing-1 have an infinite subset that computes no effective-packing-1?

That would be a "symmetric strengthening" of our result.
Compare to a classic result of Kurtz:

Theorem

Every infinite subset of a weakly 1-generic computes a weakly 1-generic.

Computably bounded subsets

Unpublished (Miller 2011)

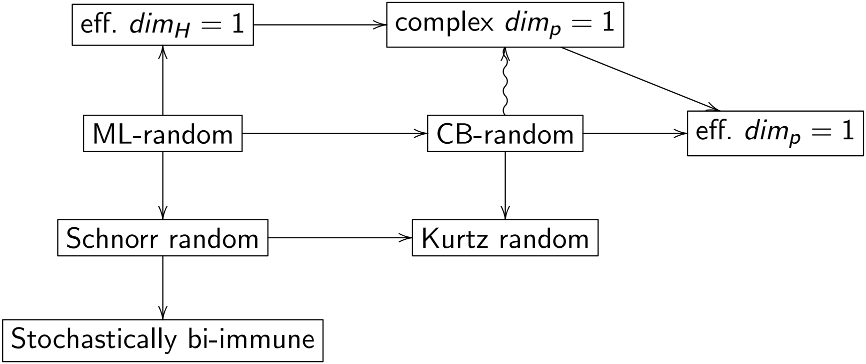
Every ML-random A has a subset B that is *computably bounded*, i.e., there is a computable bound on $n \mapsto$ the nth element of B, and computes no ML-random.

This we do not obtain in our 2019 work as we are using a variant of Mathias forcing.

Mathias generics are also *high* hence we leave the

Question

Does each infinite subset of an ML-random set compute a Schnorr random set?



$$\mathcal{A} \rightsquigarrow \mathcal{B}$$

means that each element of $\mathcal{A}$ has an infinite subset that computes no element of $\mathcal{B}$.