Constructing a weak subset of a random set
Joint work with Lu Liu

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Introduction

Liu’s approach

Complex packing dimension

Liu’s construction

CB-randomness
Reverse mathematics

- Ramsey’s Theorem for pairs
  - each 2-coloring of $[\omega]^2$ has an infinite homogeneous set
  - for each $\Delta^0_2$ definition of a possible set $A$, an infinite subset or co-subset of $A$ exists

- WKL (Weak König’s Lemma: each infinite finitely branching tree has an infinite path)

- WWKL (each tree with many branches at each level has a path)

- DNR (there is a function $f : \omega \to \omega$ with $f(e) \neq \varphi_e(e)$)
Results

- \( \text{RT}_2^2 \nvdash\text{WKL}_0 \) (Liu 2012)
- \( \text{RT}_2^2 \nvdash\text{WWKL}_0 \) (Liu 2015)
- \( \text{DNR} \nvdash\text{WWKL}_0 \) (Ambos-Spies, K., Lempp, Slaman 2004)
Corresponding computability results:

- Each $\Delta^0_2$ set has an infinite subset or co-subset that computes no completion of Peano Arithmetic (Liu 2012)
- Each $\Delta^0_2$ set has an infinite subset or co-subset that computes no ML-random set (Liu 2015)
- There is a DNR function that computes no ML-random set (Ambos-Spies, K., Lempp, Slaman 2004)
- DNR functions are Muchnik equivalent to infinite subsets of random sets (mentioned in Greenberg, J. Miller 2009)
**General question**

If $A$ has some computational power and $B$ is similar to $A$, does $B$ retain that power?

**General question**

If $A$ has some computational power and $B \subseteq A$, $B$ infinite, does $B$ retain that power?
Two examples of sets where you can recover nontrivial information from any subset:

- Very sparse sets
- Sets of prefixes \( \{ \sigma : \sigma \text{ prefix of } A \} \)
My question

AIM Workshop, Palo Alto, 2006

Does every Martin-Löf random set have an infinite subset that computes no ML-random set?

Answer: No.

Theorem (K., 2009)

Some ML-random has an infinite subset that computes no ML-random set.
It raised another question: is this typical?

**Theorem (K., 2011)**

*Almost every ML-random has an infinite subset that computes no ML-random set.*

**Theorem (K. and Liu, 2019)**

*Every ML-random has an infinite subset that computes no ML-random set.*

In this form a proof sketch (personal communication) was given by J. Miller, 2011. But we have strengthenings.
Theorem (K. and Liu, 2019)

Every ML-random has an infinite subset that computes no real of positive effective Hausdorff dimension.

Theorem (K. and Liu, 2019)

Every \textit{effectively incompressible} set has an infinite subset that computes no real of positive effective Hausdorff dimension.
But effectively incompressible $=$ CB-random studied by Brodhead, Downey, Ng. So

**Theorem (K. and Liu, 2019)**

Every CB-random has an infinite subset that computes no real of positive effective Hausdorff dimension.

Also:

**Theorem (K. and Liu, 2019)**

Every CB-random has an infinite subset that computes no real of positive complex packing dimension.
Liu’s approach

Definition (Beigel et al.)

Given a set $S \subseteq j^\omega$, an $\ell$-enumeration of $S$ is a function $g : \omega \to \text{Fin}(j^\omega)$ such that $|g(n)| \leq \ell$ and $g(n) \cap S \cap j^n \neq \emptyset$ for all $n$. A bounded enumeration of $S$ is an $\ell$-enumeration for some $\ell \in \omega$. 
Theorem

Given a set $A$ that is not effectively compressible, let $S^u \subseteq j^{<\omega}$, $u \in \omega$, be a family of co-c.e. sets such that none of the $S^u$, $u \in \omega$ admits computable bounded enumeration. Then there exists an infinite set $G \subseteq A$ such that none of the $S^u$ admits any bounded enumeration computable in $G$. 
Corollary

For any 1-random set $A$, there exists an infinite subset $G$ of $A$ such that $G$ does not compute any set with positive effective Hausdorff dimension.
The corollary follows by noting:

- the sequence of trees defining “positive effective Hausdorff dimension” does not admit a computable bounded enumeration,
- 1-randomness implies not being effectively compressible, and
- the infinite subset $G$ of $A$ gives the bounded enumeration (in fact, 1-enumeration) given by $g(n) = \{ G \upharpoonright n \}$.
The trees can be taken to be

\[ T_{m,c} = \{ \sigma \in 2^{<\omega} : (\forall k < |\sigma|) K(\sigma \upharpoonright k) \geq k/m - c \} \]

as \( X \) has positive effective Hausdorff dimension if and only if

\[ \exists m, c \quad \forall n \quad X \upharpoonright n \in T_{m,c}. \]

If \( T_{m,c} \) has a computable \( \ell \)-enumeration then we can describe \( \sigma \in T_{m,c} \) by giving \( n = |\sigma| \) and the index of \( \sigma \) in the list of up to \( \ell \) elements of \( T_{m,c} \cap 2^n \). This would show \( K(\sigma) \leq^{+} 2(\log n + \log \ell) \) which for large \( n \) contradicts \( K(\sigma) \geq |\sigma|/m - c \).
Complex packing dimension

Let the collection of all infinite computable subsets of $\omega$ be denoted by $\mathcal{C}$. Let $p : \omega \to \omega$. Let $d$ be the Hamming distance function. For $X, Y \in 2^\omega$ and $N \subseteq \omega$ we define a notion of proximity, or similarity, by

$$X \sim_{p,N} Y \iff (\exists n_0)(\forall n \in N, n \geq n_0)(d(X \upharpoonright n, Y \upharpoonright n) \leq p(n)).$$
Definition

The effective Hausdorff dimension of $A \in 2^\omega$ is

$$\dim_H(A) = \liminf_{n \in \omega} \frac{K(A \upharpoonright n)}{n}.$$ 

The complex packing dimension of $A \in 2^\omega$ is

$$\dim_{cp}(A) = \sup \inf_{N \in \mathbb{C}} \inf_{n \in \mathbb{N}} \frac{K(A \upharpoonright n)}{n}.$$ 

The effective packing dimension of $A \in 2^\omega$ is

$$\dim_p(A) = \limsup_{n \in \omega} \frac{K(A \upharpoonright n)}{n}.$$
\[0 \leq \dim_H(A) \leq \dim_{cp}(A) \leq \dim_P(A) \leq 1.\]

The inequality \(\dim_H(A) \leq \dim_{cp}(A)\) uses the fact that each cofinite set \(N \subseteq \omega\) is in \(\mathcal{C}\). The inequality \(\dim_{cp}(A) \leq \dim_P(A)\) uses the fact that each \(N \in \mathcal{C}\) is an infinite subset of \(\omega\).
Complex packing dimension


- Let $p(n) = o\left(\frac{n}{\log n}\right)$ and let $N \in \mathcal{C}$. If $X \in \text{MLR}$ and $X \sim_{p,N} Y$ then $\dim_{cp}(Y) = 1$.

- Contrast with *Dimension 1 sequences are close to randoms* by Greenberg, Miller, Shen, and Westrick (2018): Coarsely random same as effective Hausdorff dimension 1.

- Conjecture? “If $\dim_{cp}(Y) = 1$ then $\exists X \in \text{MLR}, X \sim_{p,N} Y$.”
New question

Does each real of positive complex packing dimension compute a real of positive effective Hausdorff dimension?
Corollary

For any 1-random set $A$, there exists an infinite subset of $A$, namely $G$, such that $G$ does not compute any set with positive complex packing dimension.
Proof.

This time take the trees to be

\[ T_{m,M,c} = \{ \sigma \in 2^{<\omega} : (\forall k \in M) K(\sigma \upharpoonright k) \geq k/m - c \} \]

where \( m > 0, c, m \in \omega, M \in \mathcal{C} \), and \( \mathcal{C} \) is the collection of all infinite computable sets. These trees are all co-c.e. The fact that they are not \textit{uniformly} co-c.e. is not a problem since the main theorem uses a construction that deals with each tree in forcing requirements.
Liu’s construction

We say $X \in 2^\omega$ is a $k$-partition iff

- $X = X_0 \oplus X_1 \oplus \cdots \oplus X_{k-1}$;
- $\bigcup_{i=0}^{k-1} X_i = \omega$.

A class $Q \subseteq 2^\omega$ is a $k$-partition class iff for every $X \in Q$, $X$ is a $k$-partition.
A Mathias condition is a pair $(\sigma, X)$ with $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$. We say that $(\tau, Y)$ extends the Mathias condition $(\sigma, X)$ iff $\sigma \preceq \tau$ and $Y/\tau \subseteq X/\sigma$. Write $(\tau, Y) \leq (\sigma, X)$ to denote the extension relation.

We say that a set $G$ satisfies the Mathias condition $(\sigma, X)$ if $\sigma \prec G$ and $G \subseteq X/\sigma$. 

**Definition (Mathias condition)**
Definition (Tree forcing conditions)

The forcing conditions we use to construct $G$ are tuples

$$(k, \sigma_0, \ldots, \sigma_{k-1}, Q),$$

where $k > 0$, $\sigma_i \in 2^{<\omega}$,

- $\sigma_i \subseteq A$ for all $i \leq k - 1$,
- $Q$ is a nonempty $\Pi^0_1$ $k$-partition class,
- for every $X_0 \oplus \cdots \oplus X_{k-1} \in Q$ and every $i \leq k - 1$,
  $\sigma_i \subseteq X_i \cap A$.

We regard each $X_0 \oplus \cdots \oplus X_{k-1} \in Q$ as representing $k$ many Mathias conditions $(\sigma_i, X_i), i < k$. 
Definition

We say that a condition \( d' = (k', \sigma'_0, \ldots, \sigma'_{k'-1}, Q') \) extends a condition \( d = (k, \sigma_0, \ldots, \sigma_{k-1}, Q) \), (henceforth \( d' \leq d \)), if there is a function \( f : k' \to k \) such that for all \( i < k' \),

\[
\forall Y_0 \oplus \cdots \oplus Y_{k'-1} \in Q' \; \exists X_0 \oplus \cdots \oplus X_{k-1} \in Q \; \left[ (\sigma'_i, Y_i) \leq (\sigma_{f(i)}, X_{f(i)}) \right].
\]

In this case, we say that

- \( f \) witnesses the extension \( d' \leq d \);
- part \( i \) of the condition \( d' \) refines part \( f(i) \) of the condition \( d \).
Definition

We say that a set $G$ satisfies condition $(k, \sigma_0, \ldots, \sigma_{k-1}, Q)$ iff there is an $X_0 \oplus \cdots \oplus X_{k-1} \in Q$ such that $G$ satisfies some $(\sigma_i, X_i)$. In this case, we also say that $G$ satisfies $(k, \sigma_0, \ldots, \sigma_{k-1}, Q)$ on part $i$. 
We assume that for each Turing functional $\Psi$ there exists $\ell_\Psi$ depending on $\Psi$ such that for every $X$, $\Psi^X$ is an $\ell_\Psi$-enumeration with $\Psi^X(m) \downarrow \to \Psi^X(m) \subseteq j^m$. 
For each Turing functional $\Psi$ and $u \in \omega$, we need to satisfy the requirement $R^u_\Psi$:

$$\Psi^G$$ is not an $\ell_\Psi$-enumeration of $S^u$ if $\Psi^G$ is total.  \hfill (R^u_\Psi)
Definition

We say condition $d$ forces requirement $\mathcal{R}$ on part $i$ iff every $G$ satisfying $d$ on part $i$ also satisfies requirement $\mathcal{R}$. We say condition $d$ forces requirement $\mathcal{R}$ iff it forces $\mathcal{R}$ on all parts.
Definition

We say part $i$ of condition $c = (k, \sigma_0, \ldots, \sigma_{k-1}, Q)$ is acceptable iff there exists $X_0 \oplus \cdots \oplus X_{k-1} \in Q$ such that $X_i \cap A$ is infinite, where $A$ is the given ML-random set.
A sample combinatorial idea used

Definition

A collection $V_1, \ldots, V_n$ of sets is *not $k$-dispersed* if it can be partitioned into $k$ subcollections each having nonempty intersection.
How dispersedness is used

- The sets $V_i$ are sets of strings that we may force $\Psi^G$ into and that each may or may not have high effective Hausdorff dimension.

- We split into cases depending on how dispersed the sets $V_i$ are and whether they contain complex strings.

- If they are not too dispersed and intersect $S^u$ at length $m$ then we can give a bounded enumeration of $S^u$ at length $m$, by picking one element from each intersection.

- To find something that Gauss works on, divide his department up into a small number of research groups, and select a topic common to each group. One of those topics is of interest to Gauss! (...One of those strings is highly complex.)
Dispersedness

Invited speakers at this conference, last names:

BARTOSOVA
FISCHER
FORTNOW
HASKELL
IEMHOFF
MARQUIS
PATEL
SOSKOVA

Divide them into groups where each group has a shared letter.
$V_1 = \{\text{BARTOSOVA, HASKELL, MARQUIS, PATEL, SOSKOVA}\}$
$V_2 = \{\text{FISCHER, FORTNOW, IEMHOFF}\}$
So the set of names is not 2-dispersed. This is optimal: there is no letter shared by all.
\[ V_1 = \{ \text{BARTOSOVA, HASKELL, MARQUIS, PATEL, SOSKOVA} \} \]
\[ V_2 = \{ \text{FISCHER, FORTNOW, IEMHOFF} \} \]
So the set of names is not 2-dispersed. This is optimal: there is no letter shared by all.
Actually there are even two disjoint names.
CB-randomness and effective compressibility

Definition

- A is not Martin-Löf random if for each $c$, there is an $n$ with $K(A \upharpoonright n) \leq n - c$.
- A is effectively compressible if there exists a computable function $f : \omega \rightarrow \omega$ such that $K_u(A \upharpoonright f(n)) \leq f(n) - n$. 
CB-randomness

Brodhead, Downey and Ng (2012) defined *computably bounded finite randomness*.

**Definition**

- A Martin-Löf test \( \{ U_n \}_{n \in \omega} \) is *computably bounded* if there is some total computable function \( f \) such that the cardinality of the set of strings defining \( U_n \) is at most \( f(n) \) for every \( n \).
- A set is CB-random if it passes each computably bounded Martin-Löf test.

Each CB-random set has effective packing dimension 1.
**Theorem**

*Effectively incompressible = CB-random*

One direction:
If $K(A \upharpoonright f(n)) \leq f(n) - n$ then $A$ belongs to

$$F_n := \{ B : K(B \upharpoonright f(n)) \leq f(n) - n \}$$

which is a finite c.e. set of measure at most $2^{C-n}$.
Other direction: use:

**Definition**

A set $A$ is weakly CB-random if there is no computable $f$ and uniformly c.e. sequence $F_n \subseteq \{0, 1\}^{f(n)}$ such that $\mu F_n \leq 2^{-n}$ for each $n$ and $A \in \cap_n[F_n]$. 
Theorem

Weak CB-randomness is the same as CB-randomness.

Proof.

Suppose \( \{V_e\} \) is a CB-test: \(|V_e| \leq f(e)\) and \( \mu V_e \leq 2^{-e} \) for each \( e \).
Now pick a length \( n_e \geq e + \log_2 f(e) \). Let \( R_e = \{\sigma | n_e : \sigma \in V_e\} \).

\[
[R_e] = \{\{\sigma : \sigma \in V_e, |\sigma| \leq n_e\} \cup \{\sigma | n_e : \sigma \in V_e, |\sigma| > n_e\}\}
\]

\[
\mu R_e \leq \mu V_e + f(e)2^{-n_e} \\
\leq 2^{-e} + 2^{-e} = 2^{-(e-1)}
\]

So let \( U_e = R_{e+1} \) and the \( U_e \)'s form a WCB-test with \( \cap_e U_e \supseteq \cap_e V_e \).
Effective incompressibility via Schnorr’s theorem

TFAE

- For some ML-test $U$, we have $A \in \bigcap_{n \in \omega} U_n$
- For some ML-test $U$, we have $\forall n \ \exists k \ [A \upharpoonright k] \subseteq U_n$
- For some ML-test $U$, we have $\forall c \ \exists n \ [A \upharpoonright n] \subseteq U_c$
- $\forall c \ \exists n \ K(A \upharpoonright n) \leq n - c$
Now we “effectivize” this: TFAE

**A is not CB-random** For some computable \( f \), for some ML-test \( U \),
we have \( \forall c \ [A \upharpoonright f(c)] \in U_c \)

**A is effectively compressible** For some computable \( f \),
\( \forall c \ K(A \upharpoonright f(c)) \leq f(c) - c \)

**Proof.**

Effectively compressible implies not CB-random since
\[
\mu\{A : \exists n K(A \upharpoonright n) \leq n - c\} \leq 2^{M-c}\text{ for a constant } M.
\]
Effectively incompressible implies weakly CB-random: use a bounded request set (see e.g. Nies’ book, Theorem 3.2.9).
Where CB-randomness fits in

<table>
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<tr>
<th>computable</th>
<th>comeager</th>
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<tbody>
<tr>
<td>Kurtz random</td>
<td>Schnorr random</td>
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<tr>
<td>c.e.</td>
<td>CB-random</td>
<td>ML-random</td>
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Blocks

There is an even weaker version, block-CB randomness, which suffices for our main theorem.

**Definition**

A block-CB test uses sets $U_n$ that are of the form $U_n = \{ X : X \upharpoonright I_n \in F_n \}$ where the $I_n$ are computable disjoint intervals and the $F_n$ are uniformly c.e. finite sets of cardinality at most $1/2 \cdot 2^{|I_n|}$ so that $\mu(U_n) \leq 1/2$. 
Theorem

Block-CB random (for uniformly computable sets $F$) implies Kurtz random.

Proof.

Suppose $A$ belongs to $[T]$ which has measure 0 for a computable tree $T$. Find a length $n_0$ giving measure at most $1/2$. How do we choose $n_1$ to get measure at most $1/2$ for the interval $T \cap [n_0, n_1]$? If a fraction $\geq \epsilon$ of the $2^{n_1-n_0}$ strings in $[n_0, n_1]$ appear in $T$ then the measure of $T \upharpoonright n_1$ is at least $\epsilon 2^{n_1-n_0} 2^{-n_1} = \epsilon 2^{-n_0}$. So if we take $\epsilon = 1/2$, choose $n_1$ such that $\mu T \upharpoonright n_1 < 2^{-n_0-1}$. And iteratively choose $n_{s+1}$ such that

$$\mu(T \upharpoonright n_{s+1}) < 2^{-n_{s+1}-1}.$$
Same idea shows:

**Theorem**

*Block-CB random implies CB random.*

**Proof.**

We do this by taking large blocks: If \( A \) belong to \( F_n \) which has probability at most \( 2^{-n} \) and contained in \([0, m_n]\) then take \( G_n \) to be the interval \( I = [m_n, m_f(n)] \) information in \( \bigcap_n F_n \),

\[
G_n = \{ X : (\exists Y) X \upharpoonright I = Y \upharpoonright I, Y \in \bigcap_n F_n \}
\]

where \( f \) is so large that \( \mu(F_f(n)) < 2^{-m_n-1} \), which we ensure by letting \( f(n) = m_{n+1} (\geq m_n + 1) \) say.

So the block-nature of the proof in our main theorem is not important.
Theorem (Brodhead, Downey, Ng)

A Turing incomplete c.e. real can be CB-random.

Question

Does every CB-random have an infinite subset that is still CB-random and computes no set of positive complex packing dimension?
Question
Does every effective-packing-1 have an infinite subset that computes no effective-packing-1?

That would be a “symmetric strengthening” of our result. Compare to a classic result of Kurtz:

Theorem
Every infinite subset of a weakly 1-generic computes a weakly 1-generic.
Computably bounded subsets

Unpublished (Miller 2011)

Every ML-random $A$ has a subset $B$ that is \textit{computably bounded}, i.e., there is a computable bound on $n \mapsto$ the $n$th element of $B$, and computes no ML-random.

This we do not obtain in our 2019 work as we are using a variant of Mathias forcing.
Mathias generics are also *high* hence we leave the

**Question**

Does each infinite subset of an ML-random set compute a Schnorr random set?
$A \rightsquigarrow B$

means that each element of $A$ has an infinite subset that computes no element of $B$. 