Factoring and roots

**Theorem.** If \( a > 0 \), \( x^2 - a = (x - \sqrt{a})(x + \sqrt{a}) \).

But \( x^2 + a \) has no roots and can’t be factored any more.

**Division Law.** If \( p(x)/d(x) \) has quotient \( q(x) \) and remainder \( r(x) \) then
\[
\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.\]

Multiply by \( d(x) \) to get \( p(x) = d(x)q(x) + r(x) \).

\( d(x) \) divides into \( p(x) \) evenly iff the remainder is 0 iff \( p(x) = d(x)q(x) \) iff \( d(x) \) is a factor of \( p(x) \).

If \( d(x) \) is a factor of \( p(x) \), the other factor of \( p(x) \) is \( q(x) \), the quotient of \( p(x)/d(x) \).

- **Given** \( p(x)/d(x) \), divide to get the quotient \( q(x) \) and remainder \( r(x) \). Write the answer in division law form:
  \[ p(x) = d(x)q(x) + r(x). \]

\[
\frac{x^3+1}{x+1}, \quad \frac{x^3+1}{x^2+1} = x^2 + x + 1 + \frac{2}{x+1}, \quad x^3 + 1 = (x - 1)(x^2 + x + 1) + 2
\]

Check that the answer is correct for \( x = 0 \):
For \( x = 0 \), we get \( 0 + 1 = (-1)(0 + 1) + 1 \), \( 1 = 1 \).

**X-intercept.** \( a \) is a root or zero of \( p(x) \) iff \( p(a) = 0 \).

**Theorem.** \( a \) is a root of \( p(x) \) iff \((x - a) \) is a factor of \( p(x) \). Note, rewrite \((x + 3)\) as \((x - (-3))\).

To find all roots of \( p(x) \), completely factor \( p(x) \).

**Factor the polynomial and find all roots.**

- \( x + 2 \) Root: \(-2\)
- \( x^2 + 2 \) Fully factored as is, no roots.
- \( x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}) \) Rts: \(-\sqrt{2}, \sqrt{2}\)
- \( x^2 - 4x + 4 = (x - 2)^2 \) One repeated factor. Root: \( 2 \)
- \( x^3 + 5x^2 + 8x + 4 \) given that \(-1\) is a root.
  \((x - (-1)) = (x + 1) \). we divide by \((x + 1)\).
  \[
  \frac{x^3+5x^2+8x+4}{x+1} = x^2 + 4x + 4 = (x + 2)(x + 2) = (x + 2)^2
  \]
  \[
  x^3 + 5x^2 + 8x + 4 = (x + 1)(x + 2)^2 \] Roots: \(-2, -1\).
  \[
  = (x - 1)(x + 2)^2 \] optional minus form.
- \( x^3 - x^2 - 2x + 2 \) given that \( 1 \) is a root.
  \[
  \frac{x^3-x^2-2x+2}{x-1} = x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}).
  \]
  \[
  x^3 - x^2 - 2x + 2 = (x - 1)(x + \sqrt{2})(x - \sqrt{2})
  \] Roots: \(-\sqrt{2}, \sqrt{2}\).
- \( 2x^2 + 2x - 1 \). Find the roots with the quadratic formula. There will be three factors: one for each root and one for the coefficient 2 of \( x^2 \).
  \[
  2x + 2x = \sqrt{2} + \frac{\sqrt{2} - 4ac}{2a} = \frac{2(2)^2 - 4(2)(-1)}{2(2)} = \frac{-2 + \sqrt{12}}{4} = \frac{-1 + \sqrt{3}}{2}
  \]
  Factorization: \( 2(x - \frac{1 + \sqrt{3}}{2})(x - \frac{-1 + \sqrt{3}}{2}) \)

Functions

**Definition.** For sets \( A \) and \( B \), a function from \( A \) to \( B \) assigns a value \( f(x) \) in \( B \) to each \( x \) in \( A \). The domain of \( f \) is \( A \); the range of \( f \) is the set of all possible values \( f(x) \)

- \( f(x) = x^2 \) is a function from real numbers to real numbers.
  - domain = \((-\infty, \infty)\) since \( x^2 \) is defined for all numbers. range = \([0, \infty)\) since \( x^2 \) can never be negative.

**Notation.** Sometimes, instead of writing \( f(x) = x^2 \), we define a function by writing \( y = x^2 \).

Thus \( y \) is the value of the function. Since it depends on \( x \), \( y \) is the dependent variable. Since \( x \) ranges freely over the domain, it is the independent variable.

A function may assign only one value to each \( x \).
Thus \( y = \pm \sqrt{x} \) is not a function.

- **Of** \( f \) and \( g \), which are functions? \( f \) isn’t, \( g \) is
  - \( y = 1 - x \) \((-\infty, \infty)\)
  - \( y = \frac{1}{1-x} \) \((-\infty, 1) \cup (1, \infty)\)
  - \( y = \sqrt{1-x} \) \((-\infty, 1]\)

- \( f(x) = x^2 \). First add \( ( ) \)’s around each \( x \): \((x)^2 \). Simplify to an expanded polynomial.
  - \[ \frac{f(x) - f(a)}{x-a} = \frac{(x^2 - a^2)}{x-a} = x + a \]
  - \[ \frac{f(x + h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = 2x + h \]

To get \( f(x + h) \), replace \( x \) in \( f(x) = x^2 \) by \( (x + h) \) to get \( f(x + h) = (x + h)^2 \).
Note, \( f(x + h) \neq x^2 + h^2 \).

- **g(x) = \frac{1}{x} - x.** Rewrite as \( \frac{1}{x} - (x) \). Simplify
  \[
  g(g(x)) = \frac{1}{g(x)} - g(x) = \frac{1}{(\frac{1}{x} + x)} - (\frac{1}{x} - x) = \frac{2x^2}{x(x+1)} + \frac{2x^2(x+1)}{x} = \frac{2x^3 + 2x^2}{x(x+1)}.
  \]

- **h(x) = \frac{1-x}{x}.** Simplify
  \[
  h(x + h) = \frac{1-(x+h)}{(x+h)} = \frac{1}{x} - \frac{h}{x+h}.\]
  - \[ h(h(x)) = \frac{1-(h(x))}{(h(x))} = \frac{1-(\frac{1}{x})}{(\frac{1}{x})} = \cdots = \frac{2x-1}{1-x}. \]