Definition. A theory is a set of sentences. A type is a set of formulas $\varphi(x)$ in some fixed variable x. A structure $\mathfrak{A}$ is a model of a theory if all the theory's sentences are true in $\mathfrak{A}$. A type is realized in $\mathfrak{A}$ if there is an a in $\mathfrak{A}$ which makes all the type's formulas true. A theory is consistent if it has a model. By Gödel's Completeness Theorem, this is equivalent to saying one cannot prove a contradiction from the theory. A type is finitely realized in $\mathfrak{A}$ if every finite subset is realized in $\mathfrak{A}$. In this case we allow the formulas to have parameters from $\mathfrak{A}$ (new constants which name elements of $\mathfrak{A}$).

Compactness Theorem. If every finite subset of a theory is consistent then the theory is consistent.

Definition. $\mathfrak{B}$ is saturated over $\mathfrak{A}$ if $\mathfrak{A} \preceq \mathfrak{B}$ ($\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$) and, in addition, every type with parameters from $\mathfrak{A}$ which is finitely realized in $\mathfrak{A}$ is realized in $\mathfrak{B}$.

Theorem. Every $\mathfrak{A}$ has an elementary extension $\mathfrak{B}$ such that $\mathfrak{B}$ is saturated over $\mathfrak{A}$.

EXERCISE: Suppose $\mathbf{R} = \langle R, <, +, \times, 0, 1 \rangle$ and suppose $\mathbf{R} \preceq \mathbf{R}^*$ which is saturated over $\mathbf{R}$.

Prove that $\mathbf{R}^*$ has an infinite integer: an integer $n >$ every integer in $\mathbf{R}$.

$1 < x,\ 2 < x,\ 3 < x,\ 4 < x,\ \ldots$

Prove that $\mathbf{R}^*$ has an infinitesimal element: a positive ε which is < every positive real: $0 < \varepsilon < R^+$.

$0 < x,\ x < \frac{1}{2},\ x < \frac{1}{3},\ x < \frac{1}{4},\ \ldots$

THEOREM (Hrbacek?). There is a dense linearly ordered (by inclusion $\subseteq$) collection of universes of set theory with initial universe $\langle V, \in, \varnothing, R \rangle$ such that:

- Each larger universe is a saturated elementary extension of each smaller universe.
- For any set x in the union of these universes, there is a minimal universe $V(x)$ containing this set. Every universe is the minimal universe of some set x.
- Every universe U satisfies a strong comprehension axiom: For every A, formula $\varphi(x, \overline{c})$ (A, $\overline{c}$ need not be in U), there is

$$\exists B \in U \text{ such that } \forall x \in U\,[\, x \in B \Leftrightarrow x \in A \,\&\, \varphi(x, \overline{c})\,].$$

DEFINITION Relative to any universe U,

n is infinite if $R_U^+ < n$, otherwise it is finite.

$m << n$ iff $R_{U(m)}^+ < n$.

ε is infinitesimal iff $|\varepsilon| < R_U^+$.

$\varepsilon << \delta$ iff $|\varepsilon| < R_{U(\delta)}^+$.

$r \approx s$, r, s are ultraclose, iff $|r - s|$ is infinitesimal.

PROPOSITION $U \subset V \;\&\; x \approx_V y \;\Rightarrow\; x \approx_U y$

Theorem. Every finite real is ultraclose to a standard real.

Proof. Let r^* be a finite real (presumably nonstandard).
Let r be the (standard) least upper bound of $\{\text{standard } x : x \leq r^*\}$. Then r is ultraclose to r^* since:

If r is a standard distance above r^*, then r is not the least upper bound.

If r is a standard distance below r^*, then r is not an upper bound for numbers $< r^*$.

Theorem (in Hrbacek's universe). Relative to a universe U,
(1) For every infinite n, there is an intermediate infinite m, $R^+ << m << n$.
(2) For every infinitesimal ε, there is a intermediate infinitesimal δ: $\varepsilon << \delta < R_U^+$.

Proof: (1) Use the density of universes to pick m's universe to be between U and n's universe. $U \subset U(m) \subset U(n)$.

Proposition. For any a, any real function f in any universe U:

- $\lim_{n\to\infty} a_n = b$ iff $a_N \approx b$ for all infinite N.
- f is continuous at a iff $f(x) \approx f(a)$ when $x \approx a$.

In single-level nonstandard analysis, you can use this to prove polynomials are continuous on the standard reals. But to prove they are continuous on nonstandard reals, you have to return to the ε-δ method. Here, you don't.

The next two proofs use of multiple levels of nonstandard reals. Both are from Hrbacek's "Analysis with ultrasmall numbers", American Mathematical Monthly, Nov. 2010.

THEOREM (EULER). $\lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{n=0}^{\infty} \frac{1}{n!}$. (= e)

Proof. Since the right sum convergences by the ratio test, it suffices to show that $(1+\frac{1}{N})^N \approx \sum_0^\infty \frac{1}{n!}$ for all infinite N.

Given N, pick an intermediate infinite M: $R^+ << M << N$.

$$(1+\frac{1}{N})^N = \sum_{n=0}^{N} \frac{N!}{(N-n)!\,n!} \cdot 1^{N-n} \cdot (\frac{1}{N})^n \quad \text{(binomial theorem)}$$

$$= \sum_{n=0}^{N} \frac{N\,(N-1)\,(N-2)\,\ldots\,(N-(n-1))}{n!} \frac{1}{N^n}$$

$$= \sum_{n=0}^{N} \frac{1(1-\frac{1}{N})(1-\frac{2}{N})\ldots(1-\frac{n-1}{N})}{n!}$$

$$= \sum_{n=0}^{M} \frac{1(1-\frac{1}{N})(1-\frac{2}{N})\ldots(1-\frac{n-1}{N})}{n!} \qquad \text{(call this } L\text{)}$$

$$+ \sum_{n=M+1}^{N} \frac{1(1-\frac{1}{N})(1-\frac{2}{N})\ldots(1-\frac{n-1}{N})}{n!} \qquad \text{(call this } K\text{)}$$

$$= L + K$$

Recall: $R^+ << M << N$.

Thus, in the universe of M, M is finite but N is infinite. Thus $\frac{1}{N}, \frac{2}{N}, \ldots, \frac{n-1}{N}, \frac{M}{N}$ are infinitesimal.

Hence $1-\frac{1}{N} \approx 1,\ 1-\frac{2}{N} \approx 1,\ \ldots,\ 1-\frac{n-1}{N} \approx 1$

Hence $1(1-\frac{1}{N})(1-\frac{2}{N})\ldots(1-\frac{n-1}{N}) \approx_M 1$ since it is a finite product of numbers ultraclose to 1.

Hence

$$L = \sum_{n=0}^{M} \frac{1(1-\frac{1}{N})(1-\frac{2}{N})\ldots(1-\frac{n-1}{N})}{n!} \approx_M \sum_{n=0}^{M} \frac{1}{n!} \approx_R \sum_{n=0}^{\infty} \frac{1}{n!}$$

$$K = \sum_{n=M+1}^{N} \frac{1(1-\frac{1}{N})(1-\frac{2}{N})\ldots(1-\frac{n-1}{N})}{n!} \text{ is}$$

$$\leq \sum_{n=M+1}^{N} \frac{1(1)(1)\ldots(1)}{n!} \leq \sum_{n=M+1}^{\infty} \frac{1}{n!} \approx_R 0 \text{ since } \sum_{n=0}^{\infty} \frac{1}{n!}$$

converges and M is infinite.

Hence, in the standard universe of R,

$(1+\frac{1}{N})^N = L + K \approx \sum_{n=0}^{\infty} \frac{1}{n!} + 0 = \sum_{n=0}^{\infty} \frac{1}{n!}$ for all infinite N.

Hence $\lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{n=0}^{\infty} \frac{1}{n!}$. □

L'Hospital's Rule for ∞/∞. Suppose f and g are differentiable in the deleted neighborhood of a. Suppose $L = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$.

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Proof. We must show* that for all x ultraclose to a, $\frac{f(x)}{g(x)} \approx L$. We prove the case for $a < x$.

Choose y such that $x - a << y - a << R^+$. Hence $a < x < y$. By Cauchy's mean value theorem, $\exists c \in (x, y)$ such that

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(c)}{g'(c)}.$$

Since $c \approx a$, $\frac{f'(c)}{g'(c)} \approx L$, $\therefore \frac{f(x)-f(y)}{g(x)-g(y)} \approx L$.

In y's universe, x is ultraclose to a but y is not. Thus $f(y)$ and $g(y)$ are finite but $f(x)$ and $g(x)$ are infinite (since $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$). Hence $\frac{f(y)}{f(x)}, \frac{g(y)}{g(x)}$ are ≈ 0.

Hence, for all x ultraclose to a, $L \approx$

$$\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f(x)}{g(x)} \cdot \frac{1-f(y)/f(x)}{1-g(y)/g(x)} \approx_y \frac{f(x)}{g(x)}$$

which was to be shown*.