A Variational Finite Element Discretization of Compressible Flow

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Abstract

We present a finite element variational integrator for compressible flows. The numerical scheme is derived by discretizing, in a structure preserving way, the Lie group formulation of fluid dynamics on diffeomorphism groups and the associated variational principles. Given a triangulation on the fluid domain, the discrete group of diffeomorphisms is defined as a certain subgroup of the group of linear isomorphisms of a finite element space of functions. In this setting, discrete vector fields correspond to a certain subspace of the Lie algebra of this group. This subspace is shown to be isomorphic to a Raviart-Thomas finite element space. The resulting finite element discretization corresponds to a weak form of the compressible fluid equation that doesn't seem to have been used in the finite element literature. It extends previous work done on incompressible flows and at the lowest order on compressible flows. We illustrate the conservation properties of the scheme with some numerical simulations.

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Contents

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1 Introduction

Numerical schemes that respect conservation laws and other geometric structures are of paramount importance in computational fluid dynamics, especially for problems relying on long time simulation. This is the case for geophysical fluid dynamics in the context of meteorological or climate prediction.

Schemes that preserve the geometric structures underlying the equations they discretize are known as geometric integrators [14]. One efficient way to derive geometric integrators is to exploit the variational formulation of the continuous equations and to mimic this formulation at the spatially and/or temporally discrete level. For instance, in classical mechanics, a time discretization of the Lagrangian variational formulation permits the derivation of numerical schemes, called variational integrators, that are symplectic, exhibit good energy behavior, and inherit a discrete version of Noether's theorem which guarantees the exact preservation of momenta arising from symmetries, see [17].

Geometric variational integrators for fluid dynamics were first derived in [19] for the Euler equations of a perfect fluid. These integrators exploit the viewpoint of [2] that fluid motions correspond to geodesics on the group of volume preserving diffeomorphisms of the fluid domain. The spatially discretized Euler equations emerge from an application of this principle on a finite dimensional approximation of the diffeomorphism group. The approach has been extended to various equations of incompressible fluid dynamics with advected quantities [11], rotating and stratified fluids for atmospheric and oceanic dynamics [9], reduced-order models of fluid flow [16], anelastic and pseudo-incompressible fluids on 2D irregular simplicial meshes [4], compressible fluids [3], and compressible fluids on spheres [7]. In all of the aforementioned references, the schemes that result are low-order finite difference schemes.

It was suggested in [16] that the variational discretization initiated in [19] can be generalized by letting the discrete diffeomorphism group act on finite element spaces. Such an approach was developed in [18] in the context of the ideal fluid and thus allowed for a higher order version of the method as well as an error estimate. For certain parameter choices, this high order method coincides with an $H(\mathrm{div})$-conforming finite element method studied in [13].

In the present paper we develop a finite element variational discretization of compressible fluid dynamics by exploiting the recent progresses made in [3] and [18], based on the variational method initiated in [19]. Roughly speaking, our approach is the following. Given a triangulation on the fluid domain Ω, we consider the space $V_h^r \subset L^2(\Omega)$ of polynomials of degree $\leq r$ on each simplex and define the group of discrete diffeomorphisms as a certain subgroup G_h^r of the general linear group $GL(V_h^r)$. The action of G_h^r on V_h^r is understood as a discrete version of the action by pull back on functions in $L^2(\Omega)$. As a consequence, the action of the Lie algebra $\mathfrak{g}_h^r$ on V_h^r is understood as a discrete version of the derivation along vector fields. In a similar way with [19] and [3], this interpretation naturally leads one to consider a specific subspace of $\mathfrak{g}_h^r$ consisting of Lie algebra elements that actually represent discrete vector fields. We show that this subspace is isomorphic to a Raviart-Thomas finite element space. We also define a Lie algebra-to-vector fields map, that allows a systematic definition of the semidiscrete Lagrangian for any given continuous Lagrangian. The developed setting allows us to derive the finite element scheme by applying a discrete version of the Lie group variational formulation of compressible fluids. In particular the discretization corresponds to a weak form of the compressible fluid equation that doesn't seem to have been used in the finite element literature. An incompressible version of this expression of the weak form has been used in [12] for the incompressible fluid with variable density. The setting that we develop applies in general to 2D and 3D fluid models that can be written in Euler-Poincaré form. For instance it applies to the rotating shallow water equations.

The plan of the paper is the following. In Section 2, we review the variational Lie group formulation of both incompressible and compressible fluids, by recalling the Hamilton principle on diffeomorphism groups corresponding to the Lagrangian description, and the induced Euler-Poincaré variational principle corresponding to the Eulerian formulation. We also briefly indicate how this formulation has been previously used to derive variational integrators. In Section 3 we consider the distributional derivative, deduce from it a discrete derivative acting on finite element spaces, and study its properties. In particular we show that these discrete derivatives are isomorphic to a Raviart-Thomas finite element space. In the lower order setting, the space of discrete derivatives recovers the spaces used in previous works, both for the incompressible [19] and compressible [3] cases. In Section 4, we define a map that associates to any Lie algebra element of the discrete diffeomorphism group a vector field on the fluid domain. We call such a map a Lie algebra-to-vector fields map. It is needed to define in a general way the semidiscrete Lagrangian associated to a given continuous Lagrangian. We study its properties, which are used later to write down the numerical scheme. In Section 5, we derive the numerical scheme by using the Euler-Poincaré equations on the discrete diffeomorphism group associated to the chosen finite element space. As we will explain in detail, in a similar way with the approach initiated in [19], a nonholonomic version of the Euler-Poincaré principle is used to constrain the dynamics to the space of discrete derivatives. We show that such a space must be a subspace of a Brezzi-Douglas-Marini finite element space. Finally, we illustrate the behavior of the resulting scheme in Section 6.

2 Review of variational discretizations in fluid dynamics

We begin by reviewing the variational formulation of ideal and compressible fluid flows and their variational discretization.

2.1 Incompressible flow

The Continuous Setting. As we mentioned in the introduction, solutions to the Euler equations of ideal fluid flow in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary can be formally regarded as curves $\varphi : [0,T] \to \mathrm{Diff}_{\mathrm{vol}}(\Omega)$ that are critical for the Hamilton principle

$$\delta \int_0^T L(\varphi, \partial_t \varphi) \mathrm{d}t = 0 \tag{1}$$

with respect to variations $\delta\varphi$ vanishing at the endpoints. Here $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ is the group of volume preserving diffeomorphisms of Ω and $\varphi(t) : \Omega \to \Omega$ is the map sending the position X of a fluid particle at time 0 to its position $x = \varphi(t, X)$ at time t. The Lagrangian in (1) is given by the kinetic energy

$$L(\varphi, \partial_t \varphi) = \int_\Omega \frac{1}{2} |\partial_t \varphi|^2 \, \mathrm{d}X,$$

and is invariant under the right action of $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ on itself via composition, namely,

$$L(\varphi \circ \psi, \partial_t(\varphi \circ \psi)) = L(\varphi, \partial_t \varphi), \quad \forall \psi \in \mathrm{Diff}_{\mathrm{vol}}(\Omega).$$

This symmetry is often referred to as the particle relabelling symmetry.

As a consequence of this symmetry, the variational principle (1) can be recast on the Lie algebra of $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$, which is the space $\mathfrak{X}_{\mathrm{div}}(\Omega)$ of divergence-free vector fields on Ω with vanishing normal component on $\partial\Omega$. Namely, one seeks a curve $u : [0,T] \to \mathfrak{X}_{\mathrm{div}}(\Omega)$ satisfying the critical condition

$$\delta \int_0^T \ell(u) \, \mathrm{d}t = 0, \tag{2}$$

subject to variations δu of the form

$$\delta u = \partial_t v + \pounds_u v, \quad \text{with } v : [0,T] \to \mathfrak{X}_{\mathrm{div}}(\Omega) \text{ and } v(0) = v(T) = 0,$$

where

$$\ell(u) = \int_\Omega \frac{1}{2} |u|^2 \, \mathrm{d}x,$$

and $\pounds_u v = [u, v] = u \cdot \nabla v - v \cdot \nabla u$ is the Lie derivative of the vector field v along the vector field u. This principle is obtained from the Hamilton principle (1) by using the relation $\partial_t \varphi = u \circ \varphi$ between the Lagrangian and Eulerian velocities and by computing the constrained variations of u induced by the free variations of φ. The conditions for criticality in (2) read

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= -\nabla p, \\ \operatorname{div} u &= 0, \end{aligned} \tag{3}$$

where p is a Lagrange multiplier enforcing the incompressibility constraint. The process just described for the Euler equations is valid in general for invariant Euler-Lagrange systems on arbitrary Lie groups and is known as Euler-Poincaré reduction. It plays an important role in this paper as it is used also at the discrete level to derive the numerical scheme. We refer to Appendix A for more details on the Euler-Poincaré principle and its application to incompressible flows.

The Semidiscrete Setting. The variational principle recalled above has been used to derive structure-preserving discretizations of the incompressible Euler equations [19], and various generalizations of it have been used to do the same for other equations in incompressible fluid dynamics [11, 9]. In these discretizations, the group $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ is approximated by a subgroup $\mathring{G}_h$ of the general linear group $GL(V_h)$ over a finite-dimensional vector space V_h, and extremizers of a time-discretized action functional are sought within $\mathring{G}_h$. More precisely, extremizers are sought within a subspace of the Lie algebra $\mathring{\mathfrak{g}}_h$ of $\mathring{G}_h$ after reducing by a symmetry and imposing nonholonomic constraints. This construction typically leads to schemes with good long-term conservation properties.

The use of a subgroup of the general linear group to approximate $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ is inspired by the fact that $\mathrm{Diff}_{\mathrm{vol}}(\Omega)$ acts linearly on the Lebesgue space $L^2(\Omega)$ from the right via the pullback,

$$f \cdot \varphi = f \circ \varphi, \quad f \in L^2(\Omega), \, \varphi \in \mathrm{Diff}_{\mathrm{vol}}(\Omega).$$

This action satisfies

$$f \cdot \varphi = f, \quad \text{if } f \text{ is constant,} \tag{4}$$

and it preserves the L^2-inner product $\langle f, g\rangle = \int_\Omega fg \, \mathrm{d}x$ thanks to volume-preservation:

$$\langle f \cdot \varphi, g \cdot \varphi \rangle = \langle f, g\rangle, \quad \forall f, g \in L^2(\Omega). \tag{5}$$

In the discrete setting, this action is approximated by the (right) action of $GL(V_h)$ on V_h,

$$f \cdot q = q^{-1} f, \quad f \in V_h,\, q \in GL(V_h).$$ [1] (6)

By imposing discretized versions of the properties (4) and (5), the group $\mathring{G}_h$ is taken equal to

$$\mathring{G}_h = \{q \in GL(V_h) \mid q\mathbf{1} = \mathbf{1},\, \langle qf, qg\rangle = \langle f, g\rangle,\, \forall\, f, g \in V_h\}, \tag{7}$$

where $\mathbf{1} \in V_h$ denotes a discrete representative of the constant function 1.

While the elements of $\mathring{G}_h$ are understood as discrete versions of volume preserving diffeomorphisms, elements in the Lie algebra

$$\mathring{\mathfrak{g}}_h = \{A \in \mathfrak{gl}(V_h) \mid A\mathbf{1} = 0,\, \langle Af, g\rangle + \langle f, Ag\rangle = 0,\, \forall\, f, g \in V_h\} \tag{8}$$

of $\mathring{G}_h$ are understood as discrete volume preserving vector fields[2]. Here $\mathfrak{gl}(V_h)$ denotes the Lie algebra of $GL(V_h)$, given by the space of linear endomorphisms of V_h. The linear (right) action of the Lie algebra element $A \in \mathring{\mathfrak{g}}_h$ on a discrete function $f \in V_h$, induced by the action (6) of G_h, is given by

$$f \cdot A = -Af.$$ [3] (9)

It is understood as the discrete derivative of f in the direction A.

In early incarnations of this theory, V_h is taken equal to $\mathbb{R}^N$, where N is the number of elements in a triangulation of Ω, and $\mathbf{1} \in \mathbb{R}^N$ is the vector of all ones. In this case, we have $\langle F, G\rangle = F^\mathsf{T}\Theta G$, where $\Theta \in \mathbb{R}^{N\times N}$ is a diagonal matrix whose i^{th} diagonal entry is the volume of the i^{th} element of the triangulation. Hence $\mathring{G}_h$ is simply the group of Θ-orthogonal matrices with rows summing to 1.

In more recent treatments, a finite element formulation has been adopted [18]. Namely, V_h is taken equal to a finite-dimensional subspace of $L^2(\Omega)$, with the inner product inherited from $L^2(\Omega)$, and $\mathbf{1}$ is simply the constant function 1. This is the setting that we will develop to the compressible case in the present paper.

2.2 Compressible flows

The Continuous Setting. The Lie group variational formulation recalled above generalizes to compressible flows as follows. For simplicity we consider here only the barotropic fluid, in which the internal energy is a function of the mass density only. The variational treatment of the general

[1] Note that the representation of the group diffeomorphism by pull-back on functions is naturally a *right* action ($f \mapsto f \circ \varphi$), whereas the group $GL(V_h)$ acts by matrix multiplication on the *left* ($f \mapsto qf$). This explains the use of the inverse q^{-1} on right hand side of (6).

[2] Strictly speaking, only a subspace of this Lie algebra represents discrete vector fields, as we will see in detail later.

[3] Note the minus sign due to (6), which is consistent with the fact that $f \mapsto f \cdot A$ is a *right* representation while $f \mapsto Af$ is a *left* representation.

(or baroclinic) compressible fluid is similar, see Appendix A. Consider the group $\mathrm{Diff}(\Omega)$ of all, not necessarily volume preserving, diffeomorphisms of Ω and the Lagrangian

$$L(\varphi, \partial_t\varphi, \varrho_0) = \int_\Omega \left[\frac{1}{2}\varrho_0|\partial_t\varphi|^2 - \varrho_0 e(\varrho_0/J\varphi)\right] \mathrm{d}X. \tag{10}$$

Here ϱ_0 is the mass density of the fluid in the reference configuration, $J\varphi$ is the Jacobian of the diffeomorphism φ, and e is the specific internal energy of the fluid. The equations of evolution are found as before from the Hamilton principle

$$\delta \int_0^T L(\varphi, \partial_t\varphi, \varrho_0)\mathrm{d}t = 0, \tag{11}$$

subject to arbitrary variations $\delta\varphi$ vanishing at the endpoints and where ϱ_0 is held fixed.

The main difference with the case of incompressible fluids recalled earlier is that the Lagrangian L is not invariant under the configuration Lie group $\mathrm{Diff}(\Omega)$ but only under the subgroup $\mathrm{Diff}(\Omega)_{\varrho_0} \subset \mathrm{Diff}(\Omega)$ of diffeomorphisms that preserve ϱ_0, i.e., $\mathrm{Diff}(\Omega)_{\varrho_0} = \{\varphi \in \mathrm{Diff}(\Omega) \mid (\varrho_0 \circ \varphi)J\varphi = \varrho_0\}$, namely we have

$$L(\varphi\circ\psi, \partial_t(\varphi\circ\psi), \varrho_0) = L(\varphi, \partial_t\varphi, \varrho_0), \quad \forall \psi \in \mathrm{Diff}(\Omega)_{\varrho_0}, \tag{12}$$

as it is easily seen from (10). Note that we only have $\mathrm{Diff}(\Omega)_{\varrho_0}$-invariance because we regard the Lagrangian as a map defined on the tangent bundle $T\,\mathrm{Diff}(\Omega)$, with ϱ_0 being a fixed parameter. It is in this way that the Lagrangian is treated in the Hamilton principle (11) and this is enough to obtain the Eulerian description below.[4]

As a consequence of the symmetry (12), one can associate to L the Lagrangian $\ell(u,\rho)$ in Eulerian form, as follows

$$L(\varphi, \partial_t\varphi, \varrho_0) = \ell(u,\rho), \quad \text{with } \; u = \partial_t\varphi\circ\varphi^{-1}, \quad \rho = (\varrho_0\circ\varphi^{-1})J\varphi^{-1} \tag{13}$$

and

$$\ell(u,\rho) = \int_\Omega \left[\frac{1}{2}\rho|u|^2 - \rho e(\rho)\right]\mathrm{d}x. \tag{14}$$

From the relations (13), the Hamilton principle (11) induces the variational principle

$$\delta \int_0^T \ell(u,\rho)\,\mathrm{d}t = 0, \tag{15}$$

with respect to variations δu and $\delta\rho$ of the form

$$\delta u = \partial_t v + \pounds_u v, \quad \delta\rho = -\operatorname{div}(\rho v), \quad \text{with } v : [0,T] \to \mathfrak{X}(\Omega) \text{ and } v(0) = v(T) = 0.$$

Here $\mathfrak{X}(\Omega)$ denotes the Lie algebra of $\mathrm{Diff}(\Omega)$, which consists of vector fields on Ω, with vanishing normal component on $\partial\Omega$. The conditions for criticality in (15) yield the balance of fluid momentum

$$\rho(\partial_t u + u\cdot\nabla u) = -\nabla p, \quad \text{with} \quad p = \rho^2\frac{\partial e}{\partial\rho} \tag{16}$$

while the relation $\rho = (\varrho_0\circ\varphi^{-1})J\varphi^{-1}$ yields the continuity equation

$$\partial_t\rho + \operatorname{div}(\rho u) = 0.$$

As in the case of incompressible flow, the process just described for the group $\mathrm{Diff}(\Omega)$ is a special instance of the process of Euler-Poincaré reduction. We refer to Appendix A for more details and to §5.2 for the rotating fluid in a gravitational field.

[4] Full $\mathrm{Diff}(\Omega)$-invariance can be obtained by letting $\mathrm{Diff}(\Omega)$ also act on ϱ_0, as $L(\varphi\circ\psi, \partial_t(\varphi\circ\psi), (\varrho_0\circ\psi)J\psi) = L(\varphi, \partial_t\varphi, \varrho_0)$.

The Semidiscrete Setting. A low-order semidiscrete variational setting has been described in [3] that extends the work of [19, 11, 9] to the compressible case, with a particular focus on the rotating shallow water equations. It is based on the compressible version of the discrete diffeomorphism group (7), namely

$$G_h = \{q \in GL(V_h) \mid q\mathbf{1} = \mathbf{1}\}, \tag{17}$$

whose Lie algebra is

$$\mathfrak{g}_h = \{A \in \mathfrak{gl}(V_h) \mid A\mathbf{1} = 0\}. \tag{18}$$

A nonholonomic constraint is imposed in [3] to distinguish elements of $\mathfrak{g}_h$ that actually represent discrete versions of vector fields. In this paper, we will see how this idea generalizes to the higher order setting. The representation of $\mathfrak{g}_h$ on V_h is given as before by $f \mapsto f \cdot A = -Af$ and is understood as a discrete version of the derivative in the direction A.

Notice that we denote by $\mathring{G}_h$ and G_h the subgroups of $GL(V_h)$ when the finite element space V_h is left unspecified, similarly for the corresponding Lie algebras $\mathring{\mathfrak{g}}_h$ and $\mathfrak{g}_h$. When it is chosen as the space V_h^r of polynomials of degree $\leq r$ on each simplex, we use the notations $\mathring{G}_h^r$, G_h^r for the groups and $\mathring{\mathfrak{g}}_h^r$, $\mathfrak{g}_h^r$ for the Lie algebras.

3 The distributional directional derivative and its properties

As we have recalled above, when using a subgroup of $GL(V_h)$ to discretize the diffeomorphism group, its Lie algebra $\mathfrak{g}_h$ contains the subspace of discrete vector fields. More precisely, as linear maps in $\mathfrak{g}_h \subset \mathfrak{gl}(V_h)$, these discrete vector fields act as discrete derivations on V_h. Once a vector space V_h is selected, it is thus natural to choose these discrete vector fields as distributional directional derivatives. This choice was made in [18]. In this section we recall this definition, study its properties and show that these derivations are isomorphic to a Raviart-Thomas finite element space.

Let Ω be as before the domain of the fluid, assumed to be bounded with smooth boundary. We consider the Hilbert spaces

$$\begin{aligned} H(\mathrm{div}, \Omega) &= \{u \in L^2(\Omega)^n \mid \mathrm{div}\, u \in L^2(\Omega)\} \\ H_0(\mathrm{div}, \Omega) &= \{u \in H(\mathrm{div}, \Omega) \mid u \cdot n|_{\partial\Omega} = 0\} \\ \mathring{H}(\mathrm{div}, \Omega) &= \{u \in H(\mathrm{div}, \Omega) \mid \mathrm{div}\, u = 0,\ u \cdot n|_{\partial\Omega} = 0\}. \end{aligned}$$

The notation $\mathring{H}(\mathrm{div}, \Omega)$ is sometimes used in the literature to denote the subspace of $H(\mathrm{div}, \Omega)$ with the boundary condition $u \cdot n|_{\partial\Omega} = 0$, which is here denoted $H_0(\mathrm{div}, \Omega)$.

3.1 Definition and properties

Let $\mathcal{T}_h$ be a triangulation of Ω having maximum element diameter h. We assume that $\mathcal{T}_h$ belongs to a shape-regular, quasi-uniform family of triangulations of Ω parametrized by h. That is, there exist positive constants C_1 and C_2 independent of h such that

$$\max_{K \in \mathcal{T}_h} \frac{h_K}{\rho_K} \leq C_1, \text{ and } \max_{K \in \mathcal{T}_h} \frac{h}{h_K} \leq C_2,$$

where h_K and ρ_K denote the diameter and inradius of a simplex K. For $r \geq 0$ an integer, we consider the subspace of $L^2(\Omega)$

$$V_h^r = \{f \in L^2(\Omega) \mid f|_K \in P_r(K),\ \forall K \in \mathcal{T}_h\}, \tag{19}$$

where $P_r(K)$ denotes the space of polynomials of degree $\leq r$ on a simplex K.

Definition 3.1. Given $u \in H(\mathrm{div}, \Omega)$, the ***distributional derivative in the direction*** u is the linear map $\nabla_u^{\mathrm{dist}} : L^2(\Omega) \to C_0^\infty(\Omega)'$ defined by

$$\int_\Omega (\nabla_u^{\mathrm{dist}} f) g \,\mathrm{d}x = -\int_\Omega f \,\mathrm{div}(gu)\,\mathrm{d}x, \quad \forall g \in C_0^\infty(\Omega). \tag{20}$$

When a triangulation $\mathcal{T}_h$ is fixed, $f \in V_h^r$, and $u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)^n$, $p > 2$, the distributional directional derivative (20) can be rewritten as

$$\begin{aligned} \int_\Omega (\nabla_u^{\mathrm{dist}} f) g \,\mathrm{d}x &= \sum_{K \in \mathcal{T}_h} \int_K (\nabla_u f) g \,\mathrm{d}x - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (u \cdot n) f g \,\mathrm{d}s \\ &= \sum_{K \in \mathcal{T}_h} \int_K (\nabla_u f) g \,\mathrm{d}x - \sum_{e \in \mathcal{E}_h^0} \int_e u \cdot [\![f]\!] g \,\mathrm{d}s, \end{aligned} \tag{21}$$

for all $g \in C_0^\infty(\Omega)$. Here, $\nabla_u f = u \cdot \nabla f$ denotes the derivative of f along u, $\mathcal{E}_h^0$ denotes the set of interior $(n-1)$-simplices in $\mathcal{T}_h$ (edges in two dimensions), and $[\![f]\!]$ is defined by

$$[\![f]\!] := f_1 n_1 + f_2 n_2, \quad \text{on} \quad e = K_1 \cap K_2 \in \mathcal{E}_h^0,$$

with $f_i := f|_{K_i}$, n_1 the normal vector to e pointing from K_1 to K_2, and similarly for n_2.

Note that there are some subtleties that arise when looking at traces on subsets of the boundary if the trace is a distribution, which explains why we need to take $u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)^n$, for some $p > 2$ when passing to the second line in (21). The trace of a vector field $u \in H(\mathrm{div}, K)$ on ∂K satisfies $u \cdot n \in H^{-1/2}(\partial K) = H_0^{1/2}(\partial K)' = H^{1/2}(\partial K)'$, but the trace of u on $e \subset \partial K$ satisfies $u \cdot n \in H_{00}^{1/2}(e)'$, where $H_{00}^{1/2}(e)$ defined by

$$H_{00}^{1/2}(e) = \{ g \in H^{1/2}(e) \mid \text{ the zero-extension of } g \text{ to } \partial K \text{ belongs to } H^{1/2}(\partial K)\} \subsetneq H_0^{1/2}(e),$$

see, e.g., [5]. So for $u \in H(\mathrm{div}, \Omega)$ and smooth g, $\int_{\partial K} (u \cdot n) g \,\mathrm{d}s$ is always well-defined, but $\int_e (u \cdot n) g \,\mathrm{d}s$ need not be; some extra regularity for u is required to make it well-defined.

Definition 3.2. Given $A \in \mathfrak{gl}(V_h^r)$ and $u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)^n$, $p > 2$, we say that A ***approximates*** $-u$ ***in*** V_h^r[5] if whenever $f \in L^2(\Omega)$ and $f_h \in V_h^r$ is a sequence satisfying $\|f - f_h\|_{L^2(\Omega)} \to 0$, we have

$$\langle A f_h - \nabla_u^{\mathrm{dist}} f, g \rangle \to 0, \quad \forall g \in C_0^\infty(\Omega). \tag{22}$$

In other words, we require that A is a consistent approximation of ∇_u^{dist} in V_h^r.

Note that the above definition abuses notation slightly; we are really dealing with a sequence of A's parametrized by h.

[5]The fact that A approximates $-u$ and not u is consistent with the fact that $f \mapsto Af$ is a left Lie algebra action whereas the derivative $f \mapsto \nabla_u f$ is a right Lie algebra action.

Proposition 3.1. *Given $u \in H_0(\mathrm{div},\Omega)\cap L^p(\Omega)^n$ and $r \geq 0$ an integer, a consistent approximation of ∇_u^{dist} in V_h^r is obtained by setting $A = A_u \in \mathfrak{gl}(V_h^r)$ defined by*

$$\langle A_u f, g\rangle := \sum_{K\in\mathcal{T}_h}\int_K (\nabla_u f) g \,\mathrm{d}x - \sum_{e\in\mathcal{E}_h^0}\int_e u\cdot [\![f]\!]\{g\}\,\mathrm{d}s, \quad \forall f,g\in V_h^r, \tag{23}$$

where $\{g\} := \frac{1}{2}(g_1+g_2)$ on $e = K_1\cap K_2$.

Moreover, if $r\geq 1$, $p=\infty$, and $A\in\mathfrak{gl}(V_h^r)$ is any other operator that approximates u in V_h^r, then A must be close to A_u in the following sense: if $f\in L^2(\Omega)$, $g\in C_0^\infty(\Omega)$, and if $f_h, g_h \in V_h^r$ satisfy $\|f_h - f\|_{L^2(\Omega)}\to 0$ and $h^{-1}\|g-g_h\|_{L^2(\Omega)} + \left(\sum_{K\in\mathcal{T}_h}|g-g_h|^2_{H^1(K)}\right)^{1/2}\to 0$, then

$$\langle (A - A_u) f_h, g_h\rangle \to 0,$$

provided that $\|Af_h\|_{L^2(\Omega)} \leq C(u,f)h^{-1}$ for some constant $C(u,f)$.

As we will see in §3.3, for $r=0$, the definition of A_u in (23) recovers the one used in [3]. There, a different definition of "A approximates $-u$" than (22) was considered for the particular case $r=0$. When this definition is used, analogous statements of both parts of Proposition 3.1 hold for $r=0$, under an additional assumption on the family of meshes [3, Lemma 2.2].

Proof. The operator A_u is a consistent approximation of ∇_u^{dist}, since for all $g\in C_0^\infty(\Omega)$

$$\begin{aligned}\langle A_u f_h - \nabla_u^{\mathrm{dist}} f, g\rangle &= \langle \nabla_u^{\mathrm{dist}}(f_h - f), g\rangle \\ &= -\int_\Omega (f_h - f)\,\mathrm{div}(gu)\mathrm{d}x \\ &\leq \|f_h - f\|_{L^2(\Omega)}\|u\cdot\nabla g + g\,\mathrm{div}\, u\|_{L^2(\Omega)} \ \to\ 0.\end{aligned}$$

For the second part, we note that

$$\begin{aligned}\langle Af_h, g_h\rangle - \langle A_u f_h, g_h\rangle &= \langle Af_h - A_u f_h, g\rangle + \langle Af_h, g_h - g\rangle - \langle A_u f_h, g_h - g\rangle \\ &= \langle Af_h - \nabla_u^{\mathrm{dist}} f_h, g\rangle + \langle Af_h, g_h - g\rangle - \langle A_u f_h, g_h - g\rangle \\ &= \langle Af_h - \nabla_u^{\mathrm{dist}} f, g\rangle + \langle \nabla_u^{\mathrm{dist}}(f - f_h), g\rangle + \langle Af_h, g_h - g\rangle - \langle A_u f_h, g_h - g\rangle \\ &\leq |\langle Af_h - \nabla_u^{\mathrm{dist}} f, g\rangle| + \|f - f_h\|_{L^2(\Omega)}\|\,\mathrm{div}(ug)\|_{L^2(\Omega)} \\ &\quad + \|Af_h\|_{L^2(\Omega)}\|g_h - g\|_{L^2(\Omega)} + |\langle A_u f_h, g_h - g\rangle|.\end{aligned}$$

By assumption, the first three terms above tend to zero as $h\to 0$. The last term satisfies

$$\begin{aligned}\langle A_u f_h, g_h - g\rangle &= \langle A_u f_h, g_h\rangle - \langle \nabla_u^{\mathrm{dist}} f_h, g\rangle \\ &= \sum_{K\in\mathcal{T}_h}\int_K (\nabla_u f_h)(g_h - g)\,\mathrm{d}x - \sum_{e\in\mathcal{E}_h^0}\int_e u\cdot[\![f_h]\!]\{g_h - g\}\,\mathrm{d}s\end{aligned}$$

since $\{g\} = g$ on each $e\in\mathcal{E}_h^0$. To analyze these integrals, we make use of the inverse estimate [10]

$$|f_h|_{H^1(K)} \leq C h_K^{-1}\|f_h\|_{L^2(K)}, \quad \forall f_h\in V_h^r, \forall K\in\mathcal{T}_h,$$

and the trace inequality [1]

$$\|f\|_{L^2(\partial K)} \leq C\left(h_K^{-1/2}\|f\|_{L^2(K)} + h_K^{1/2}|f|_{H^1(K)}\right), \quad \forall f\in H^1(K), \forall K\in\mathcal{T}_h.$$

Using the inverse estimate, we see that

$$\begin{aligned}\left|\int_K (\nabla_u f_h)(g_h - g)\,\mathrm{d}x\right| &\le \|u\|_{L^\infty(\Omega)}|f_h|_{H^1(K)}\|g_h - g\|_{L^2(K)}\\ &\le Ch_K^{-1}\|u\|_{L^\infty(\Omega)}\|f_h\|_{L^2(K)}\|g_h - g\|_{L^2(K)}.\end{aligned}$$

Using the trace inequality and the inverse estimate, we see also that

$$\begin{aligned}&\left|\int_e u\cdot [\![f_h]\!]\{g_h - g\}\,\mathrm{d}s\right|\\ &\le \frac{1}{2}\|u\|_{L^\infty(\Omega)}\left(\|f_{h1}\|_{L^2(e)} + \|f_{h2}\|_{L^2(e)}\right)\left(\|g_{h1} - g_1\|_{L^2(e)} + \|g_{h2} - g_2\|_{L^2(e)}\right)\\ &\le C\|u\|_{L^\infty(\Omega)}\left(h_{K_1}^{-1/2}\|f_h\|_{L^2(K_1)} + h_{K_2}^{-1/2}\|f_h\|_{L^2(K_2)}\right)\\ &\quad\times\left(h_{K_1}^{-1/2}\|g_h - g\|_{L^2(K_1)} + h_{K_1}^{1/2}|g_h - g|_{H^1(K_1)} + h_{K_2}^{-1/2}\|g_h - g\|_{L^2(K_2)} + h_{K_2}^{1/2}|g_h - g|_{H^1(K_2)}\right),\end{aligned}$$

where $K_1, K_2 \in \mathcal{T}_h$ are such that $e = K_1 \cap K_2$, $f_{hi} = f_h|_{K_i}$, $g_{hi} = g_h|_{K_i}$, and $g_i = g|_{K_i}$. Summing over all $K \in \mathcal{T}_h$ and all $e \in \mathcal{E}_h^0$, and using the quasi-uniformity of $\mathcal{T}_h$, we get

$$|\langle A_u f_h, g_h - g\rangle| \le C\|u\|_{L^\infty(\Omega)}\|f_h\|_{L^2(\Omega)}\left(h^{-1}\|g_h - g\|_{L^2(\Omega)} + \Big(\sum_{K\in\mathcal{T}_h}|g - g_h|^2_{H^1(K)}\Big)^{1/2}\right) \to 0.$$

□

Note that formula (23) for A_u is obtained from formula (21), valid for $f \in V_h^r$ and $g \in C_0^\infty(\Omega)$, by rewriting it for the case where $g \in V_h^r$ and choosing to replace $g \to \{g\}$ in the second term in (21).

Proposition 3.2. *For all $u \in H_0(\mathrm{div},\Omega) \cap L^p(\Omega)$, $p > 2$, we have*

$$A_u\mathbf{1} = 0 \quad \textit{and} \quad \langle A_u f, g\rangle + \langle f, A_u g\rangle + \langle f, (\operatorname{div} u)g\rangle = 0, \quad \forall f, g \in V_h^r. \tag{24}$$

Hence, if $u \in \mathring{H}(\mathrm{div},\Omega)$, then

$$A_u\mathbf{1} = 0 \quad \textit{and} \quad \langle A_u f, g\rangle + \langle f, A_u g\rangle = 0, \quad \forall f, g \in V_h^r. \tag{25}$$

Proof. The first property $A_u\mathbf{1} = 0$ follows trivially from the expression (23) since $\nabla_u 1 = 0$ and $[\![1]\!] = 0$. We now prove the second equality. Using (23) and $[\![fg]\!] = [\![f]\!]\{g\} + [\![g]\!]\{f\}$, we compute

$$\begin{aligned}&\langle A_u f, g\rangle + \langle f, A_u g\rangle\\ &= \sum_{K\in\mathcal{T}_h}\int_K ((\nabla_u f)g + (\nabla_u g)f)\,\mathrm{d}x - \sum_{e\in\mathcal{E}_h^0}\int_e u\cdot([\![f]\!]\{g\} + [\![g]\!]\{f\})\,\mathrm{d}s\\ &= \sum_{K\in\mathcal{T}_h}\int_K \nabla_u(fg)\,\mathrm{d}x - \sum_{e\in\mathcal{E}_h^0}\int_e u\cdot[\![fg]\!]\,\mathrm{d}s\\ &= \sum_{K\in\mathcal{T}_h}\left(\int_K \nabla_u(fg)\,\mathrm{d}x - \int_{\partial K}(u\cdot n)fg\,\mathrm{d}s\right)\\ &= -\sum_{K\in\mathcal{T}_h}\int_K (\operatorname{div} u)fg\,\mathrm{d}x\\ &= -\int_\Omega (\operatorname{div} u)fg\,\mathrm{d}x.\end{aligned}$$

□

From the previous result, we get a well-defined linear map

$$\mathsf{A} : H_0(\mathrm{div},\Omega)\cap L^p(\Omega)^n \to \mathfrak{g}_h^r \subset \mathfrak{gl}(V_h^r), \quad u\mapsto \mathsf{A}(u)=A_u, \quad p>2, \tag{26}$$

with values in the Lie algebra $\mathfrak{g}_h^r = \{A\in\mathfrak{gl}(V_h^r) \mid A\mathbf{1}=0\}$ of G_h. In the divergence free case, it restricts to

$$\mathsf{A} : \mathring{H}(\mathrm{div},\Omega)\cap L^p(\Omega)^n \to \mathring{\mathfrak{g}}_h^r \subset \mathfrak{gl}(V_h^r),$$

with $\mathring{\mathfrak{g}}_h^r = \{A\in\mathfrak{gl}(V_h^r) \mid A\mathbf{1}=0,\ \langle Af,g\rangle+\langle f,Ag\rangle=0,\ \forall f,g\in V_h^r\}$ the Lie algebra of $\mathring{G}_h$, see (7), which recovers the map considered in [18].

We have defined the discrete diffeomorphism group by considering its action on discrete functions. Alternatively, it is also possible to define it by considering the action on discrete densities, see Remark 5.2.

3.2 Relation with Raviart-Thomas finite element spaces

Definition 3.3. For $r\geq 0$ an integer, we define the subspace $S_h^r\subset\mathfrak{g}_h^r\subset\mathfrak{gl}(V_h^r)$ as

$$S_h^r := \mathrm{Im}\,\mathsf{A} = \{A_u\in\mathfrak{gl}(V_h^r) \mid u\in H_0(\mathrm{div},\Omega)\}.$$

Proposition 3.3. *Let $r\geq 0$ be an integer. The space $S_h^r\subset\mathfrak{g}_h^r$ is isomorphic to the Raviart-Thomas space of order $2r$*

$$RT_{2r}(\mathcal{T}_h) = \{u\in H_0(\mathrm{div},\Omega) \mid u|_K\in (P_{2r}(K))^n + xP_{2r}(K),\ \forall K\in\mathcal{T}_h\}.$$

An isomorphism is given by $u\in RT_{2r}(\mathcal{T}_h)\mapsto A_u\in S_h^r$.

Its inverse is given by

$$A\in S_h^r \mapsto u = \sum_{K\in\mathcal{T}_h}\sum_{\alpha}\boldsymbol{\phi}_K^\alpha\sum_j\langle Af_K^{\alpha,j},g_K^{\alpha,j}\rangle + \sum_{e\in\mathcal{E}_h^0}\sum_\beta\boldsymbol{\phi}_e^\beta\sum_j\langle Af_e^{\beta,j},g_e^{\beta,j}\rangle \in RT_{2r}(\mathcal{T}_h). \tag{27}$$

In this formula:

- *$(f_K^{\alpha,j},g_K^{\alpha,j}),(f_e^{\beta,j},g_e^{\beta,j})\in V_h^r\times V_h^r$ are such that the images of $\sum_j(f_K^{\alpha,j},g_K^{\alpha,j})\in V_h^r\otimes V_h^r$ and $\sum_j(f_e^{\beta,j},g_e^{\beta,j})\in V_h^r\otimes V_h^r$ under the map*

$$V_h^r\otimes V_h^r\to RT_{2r}(\mathcal{T}_h)^*, \quad f\otimes g\longmapsto \sum_{K\in\mathcal{T}_h}(\nabla f\, g)|_K + \sum_{e\in\mathcal{E}_h^0}[\![f]\!]_e\{g\}_e \tag{28}$$

  are $(\mathbf{p}_K^\alpha,0)$ and $(0,p_e^\beta)$, respectively, which is a basis of the dual space $RT_{2r}(\mathcal{T}_h)^*$ adapted to the decomposition

$$RT_{2r}(\mathcal{T}_h)^* = \sum_{K\in\mathcal{T}_h}P_{2r-1}(K)^n \oplus \sum_{e\in\mathcal{E}_h^0}P_{2r}(e);$$

- *$\boldsymbol{\phi}_K^\alpha$, $\boldsymbol{\phi}_e^\beta$ is a basis of $RT_{2r}(\mathcal{T}_h)$ dual to the basis $\mathbf{p}_K^\alpha$ and p_e^β of $RT_{2r}(\mathcal{T}_h)^*$.*

Proof. Let us consider the linear map $\mathsf{A} : H_0(\mathrm{div},\Omega)\cap L^p(\Omega)^n\to\mathfrak{gl}(V_h^r)$ defined in (26). From a general result of linear algebra, we have $\dim(\mathrm{Im}\,\mathsf{A}) = \dim(\mathrm{Im}\,\mathsf{A}^*)$, where $\mathsf{A}^* : \mathfrak{gl}(V_h^r)^*\to (H_0(\mathrm{div},\Omega)\cap L^p(\Omega)^n)^*$ is the adjoint to A. We have

$$\mathrm{Im}\,\mathsf{A}^* = \left\{\sum_{i=1}^N c_i\sigma_{f_ig_i}\in (H_0(\mathrm{div},\Omega)\cap L^p(\Omega)^n)^* \;\middle|\; N\in\mathbb{N},\ f_i,g_i\in V_h^r,\ c_i\in\mathbb{R},\ i=1,2,\ldots,N\right\},$$

where the linear form $\sigma_{fg} := \mathsf{A}^*(f \otimes g) \colon H_0(\operatorname{div}, \Omega) \cap L^p(\Omega)^n \to \mathbb{R}$, is given by $\sigma_{fg}(u) = \langle f, A_u g\rangle$.

Now, the space $\operatorname{Im} \mathsf{A}^*$ is spanned by functionals of the form

$$u \mapsto \int_e (u \cdot n) pq \, \mathrm{d}s, \quad p, q \in P_r(e),\, e \in \mathcal{E}_h^0, \tag{29}$$

and

$$u \mapsto \int_K (u \cdot \nabla q) p \, \mathrm{d}x \quad p, q \in P_r(K),\, K \in \mathcal{T}_h. \tag{30}$$

This can be seen by choosing appropriate f and g in σ_{fg} and using the definition (23) of A_u. Indeed, if we choose two adjacent simplices K_1 and K_2 and set $f|_{K_1} = p \in P_r(K_1)$, $g|_{K_2} = -2q \in P_r(K_2)$, $f|_{\Omega \setminus K_1} = 0$, and $g|_{\Omega\setminus K_2} = 0$, we get $\langle f, A_u g\rangle = \int_e (u \cdot n) pq \, \mathrm{d}s$ with $e = K_1 \cap K_2$. Likewise, if we choose a simplex K and set $f|_K = p \in P_r(K)$, $g|_K = q \in P_r(K)$, and $f|_{\Omega\setminus K} = g|_{\Omega \setminus K} = 0$, we get $\langle f, A_u g \rangle = \int_K (u \cdot \nabla q) p \, \mathrm{d}x - \frac{1}{2} \int_{\partial K} (u \cdot n) pq \, \mathrm{d}s$. Taking appropriate linear combinations yields the functionals (29-30).

Now observe that the functionals (29-30) span the same space (see Lemmas C.1 and C.2 in Appendix C) as the functionals

$$u \mapsto \int_e (u \cdot n) p \, \mathrm{d}s, \quad p \in P_{2r}(e),\, e \in \mathcal{E}_h^0,$$

and

$$u \mapsto \int_K u \cdot p \, \mathrm{d}x \quad p \in P_{2r-1}(K)^n,\, K \in \mathcal{T}_h.$$

These functionals are well-known [8]: they are a basis for the dual of

$$RT_{2r}(\mathcal{T}_h) = \{u \in H(\operatorname{div}, \Omega) \mid u \cdot n|_{\partial\Omega} = 0,\ u|_K \in (P_{2r}(K))^n + x P_{2r}(K),\, \forall K \in \mathcal{T}_h\},$$

often referred to as the "degrees of freedom" for $RT_{2r}(\mathcal{T}_h)$.

We thus have proven that $\operatorname{Im} \mathsf{A}^*$ is isomorphic to $RT_{2r}(\mathcal{T}_h)^*$, and hence $S_h^r = \operatorname{Im} \mathsf{A}$ is isomorphic to $RT_{2r}(\mathcal{T}_h)$, all these spaces having the same dimensions. Now, let us consider the linear map $u \in RT_{2r}(\mathcal{T}_h) \to \mathsf{A}(u) = A_u \in S_h^r$. Since its kernel is zero, the map is an isomorphism.

Consider now a basis $\mathbf{p}_K^\alpha$, p_e^β of the dual space $RT_{2r}(\mathcal{T}_h)^*$ identified with $\sum_{K \in \mathcal{T}_h} P_{2r-1}(K)^n \oplus \sum_{e \in \mathcal{E}_h^0} P_{2r}(e)$, i.e., the collection $\{\mathbf{p}_K^\alpha\}$ is a basis of $\sum_{K\in\mathcal{T}_h} P_{2r-1}(K)^n$ and the collection $\{p_e^\beta\}$ is a basis of $\sum_{e \in \mathcal{E}_h^0} P_{2r}(e)$. There is a dual basis $\boldsymbol{\phi}_K^\alpha$, $\boldsymbol{\phi}_e^\beta$ of $RT_{2r}(\mathcal{T}_h)$ such that

$$\langle (\mathbf{p}_K^\alpha, 0), \boldsymbol{\phi}_{K'}^{\alpha'} \rangle = \delta_{\alpha\alpha'} \delta_{KK'} \qquad \langle (\mathbf{p}_K^\alpha, 0), \boldsymbol{\phi}_e^\beta \rangle = 0$$

$$\langle (0, p_e^\beta), \boldsymbol{\phi}_{e'}^{\beta'} \rangle = \delta_{\beta\beta'} \delta_{ee'} \qquad \langle (0, p_e^\beta), \boldsymbol{\phi}_K^\alpha \rangle = 0,$$

where

$$\langle (\mathbf{p}, p), u \rangle = \sum_{K \in \mathcal{T}_h} \int_K u \cdot \mathbf{p} \, \mathrm{d}x + \sum_{e \in \mathcal{E}_h^0} \int_e (u \cdot n) p \, \mathrm{d}s$$

is the duality pairing between $RT_{2r}(\mathcal{T}_h)^*$ and $RT_{2r}(\mathcal{T}_h)$.

Choosing $f_K^{\alpha,j}, g_K^{\alpha,j} \in V_h^r$ and $f_e^{\beta,j}, g_e^{\beta,j} \in V_h^r$ such that

$$\sum_j \nabla f_K^{\alpha,j} g_K^{\alpha,j}|_{K'} = \mathbf{p}_K^\alpha \delta_{KK'}, \qquad \sum_j [\![f_K^{\alpha,j}]\!]_e \{ g_K^{\alpha,j} \}_e = 0,$$

$$\sum_j \nabla f_e^{\beta,j} g_e^{\beta,j}|_K = 0, \qquad \sum_j [\![f_e^{\beta,j}]\!]_{e'} \{ g_e^{\beta,j} \}_{e'} = p_e^{\beta} \delta_{ee'},$$

we have that

$$\sum_j \langle A_u f_K^{\alpha,j}, g_K^{\alpha,j} \rangle \quad \text{and} \quad \sum_j \langle A_u f_e^{\beta,j}, g_e^{\beta,j} \rangle$$

are exactly the degrees of freedom of u relative to the basis $\mathbf{p}_K^{\alpha}$, p_e^{β}. Therefore, u is expressed as

$$u = \sum_{K \in \mathcal{T}_h} \sum_{\alpha} \phi_K^{\alpha} \sum_j \langle A_u f_K^{\alpha,j}, g_K^{\alpha,j} \rangle + \sum_{e \in \mathcal{E}_h^0} \sum_{\beta} \phi_e^{\beta} \sum_j \langle A_u f_e^{\beta,j}, g_e^{\beta,j} \rangle$$

as desired. □

Note that the map (28), with $V_h^r \otimes V_h^r$ identified with $\mathfrak{gl}(V_h^r)^*$ can be identified with the composition $I^* \circ \mathsf{A}^*$, where I^* is the dual map to the inclusion $I : RT_{2r}(\mathcal{T}_h) \to H_0(\operatorname{div}, \Omega) \cap L^p(\Omega)^n$. This map is surjective, from the preceding result.

Proposition 3.4. *The kernel of the map $u \in H_0(\operatorname{div}, \Omega) \cap L^p(\Omega)^n \mapsto \mathsf{A}(u) = A_u \in \mathfrak{gl}(V_h^r)$, $p > 2$, is*

$$\ker \mathsf{A} = \{ u \in H_0(\operatorname{div}, \Omega) \cap L^p(\Omega) \mid \Pi_{2r}(u) = 0 \} = \ker \Pi_{2r},$$

where $\Pi_{2r} : H_0(\operatorname{div}, \Omega) \cap L^p(\Omega)^n \to RT_{2r}(\mathcal{T}_h)$ is the global interpolation operator defined by $\Pi_{2r}(v)|_K := \Pi_{2r}^K(v|_K)$, with $\Pi_{2r}^K : H(\operatorname{div}, K) \cap L^p(K)^n \to RT_{2r}(K)$ defined by the two conditions

$$\int_e \big((u - \Pi_{2r}^K u) \cdot n \big) p \, \mathrm{d}s = 0, \quad \textit{for all } p \in P_{2r}(e), \textit{ for all } e \in K$$

and

$$\int_K (u - \Pi_{2r}^K u) \cdot p \, \mathrm{d}x = 0, \quad \textit{for all } p \in P_{2r-1}(K)^n.$$

Proof. For $u \in H_0(\operatorname{div}, \Omega) \cap L^p(\Omega)^n$ we have $A_u = 0$ if and only if $\langle A_u f, g \rangle = 0$ for all $f, g \in V_h^r$. As we just commented above, the map (28) is surjective, hence from (23) we see that $\langle A_u f, g \rangle = 0$ for all $f, g \in V_h^r$ holds if and only if

$$\int_e (u \cdot n) p \, \mathrm{d}s = 0, \quad \text{for all } \ p \in P_{2r}(e), \, e \in \mathcal{E}_h^0,$$

and

$$\int_K u \cdot p \, \mathrm{d}x = 0, \quad \text{for all } \ p \in P_{2r-1}(K)^n, \, K \in \mathcal{T}_h.$$

This holds if and only if $\Pi_{2r}(u) = 0$. □

In particular for $u \in H_0(\operatorname{div}, \Omega) \cap L^p(\Omega)$, $p > 2$, there exists a unique $\bar{u} \in RT_{2r}(\mathcal{T}_h)$ such that $A_{\bar{u}} = A_u$. It is given by $\bar{u} = \Pi_{2r}(u)$.

3.3 The lowest-order setting

We now investigate the setting in which $r = 0$ in order to connect with the previous works [19] and [3] for both the incompressible and compressible cases. Enumerate the elements of $\mathcal{T}_h$ arbitrarily from 1 to N, and let $\{\psi_i\}_i$ be the orthogonal basis for V_h^0 given by

$$\psi_i(x) = \begin{cases} 1, & \text{if } x \in K_i, \\ 0, & \text{otherwise}, \end{cases}$$

where $K_i \in \mathcal{T}_h$ denotes the i^{th} element of $\mathcal{T}_h$. Relative to this basis, for $A \in \mathfrak{g}_h^0 \subset \mathfrak{gl}(V_h^0)$ we have

$$A\Big(\sum_{j=1}^N f_j \psi_j\Big) = \sum_{i=1}^N \Big(\sum_{j=1}^N A_{ij} f_j\Big)\psi_i, \quad \forall f = \sum_{j=1}^N f_j \psi_j \in V_h,$$

where

$$A_{ij} = \frac{\langle \psi_i, A\psi_j\rangle}{\langle \psi_i, \psi_i\rangle} = \frac{1}{|K_i|}\langle \psi_i, A\psi_j\rangle. \tag{31}$$

In what follows, we will abuse notation by writing A for both the operator $A \in \mathfrak{g}_h^0$ and the matrix $A \in \mathbb{R}^{N\times N}$ with entries (31). It is immediate from (8) that

$$\mathring{\mathfrak{g}}_h^0 = \Big\{A \in \mathbb{R}^{N\times N} \;\Big|\; \sum_{j=1}^N A_{ij} = 0, \forall i, \text{ and } A^\mathsf{T}\Theta + \Theta A = 0\Big\}, \quad \mathfrak{g}_h^0 = \Big\{A \in \mathbb{R}^{N\times N} \;\Big|\; \sum_{j=1}^N A_{ij} = 0\Big\}$$

where Θ is a diagonal $N \times N$ matrix with diagonal entries $\Theta_{ii} = |K_i|$. These are the Lie algebras used in [19] and [3].

The next lemma determines the subspace $S_h^0 := \operatorname{Im} \mathsf{A}$ in the case $r = 0$. We write $j \in N(i)$ to indicate that $j \neq i$ and $K_i \cap K_j$ is a shared $(n-1)$-dimensional simplex.

Lemma 3.5. *If $A = -A_u$ for some $u \in H_0(\operatorname{div}, \Omega) \cap L^p(\Omega)^n$, $p > 2$, then, for each i,*

$$\begin{aligned} A_{ij} &= -\frac{1}{2|K_i|}\int_{K_i\cap K_j} u \cdot n \,\mathrm{d}s, \quad j \in N(i), \\ A_{ii} &= \frac{1}{2|K_i|}\int_{K_i} \operatorname{div} u \,\mathrm{d}x, \end{aligned} \tag{32}$$

and $A_{ij} = 0$ for all other j.

Proof. Let $j \in N(i)$ and consider the expression (23) with $f = \psi_j$ and $g = \psi_i$. All terms vanish except one, giving

$$\begin{aligned} \langle A\psi_j, \psi_i\rangle &= \int_{K_i\cap K_j} u \cdot [\![\psi_j]\!]\{\psi_i\}\,\mathrm{d}s \\ &= -\frac{1}{2}\int_{K_i\cap K_j} u \cdot n \,\mathrm{d}s. \end{aligned}$$

Now consider the case in which $i = j$. Let $\mathcal{E}^0(K_i)$ denote the set of $(n-1)$-simplices that are on the boundary of K_i but in the interior of Ω. Since $u \cdot n = 0$ on $\partial\Omega$,

$$\begin{aligned}
\langle A\psi_i, \psi_i \rangle &= \sum_{e \in \mathcal{E}^0(K_i)} \int_e u \cdot [\![\psi_i]\!] \{\psi_i\} \, \mathrm{d}s, \\
&= \int_{\partial K_i} u \cdot n \frac{1}{2} \, \mathrm{d}s \\
&= \frac{1}{2} \int_{K_i} \operatorname{div} u \, \mathrm{d}x.
\end{aligned}$$

The expressions in (32) follow from (31). Finally, if $j \neq i$ and $j \notin N(i)$, then all terms in (23) vanish when $f = \psi_j$ and $g = \psi_i$. $\square$

Remark 3.1. The expressions in (32) recover the relations between Lie algebra elements and vector fields used in [19] and [3]. In particular, in the incompressible case, using also (25) in Proposition 3.2, we have

$$\operatorname{Im} \mathsf{A} = \left\{ A \in \mathring{\mathfrak{g}}_h^0 \mid A_{ij} = 0, \ \forall j \notin N(i) \right\}$$

which is the nonholonomic constraint used in [19]. Similarly, in the compressible case, using (24) in Proposition 3.2, we have

$$\operatorname{Im} \mathsf{A} = \left\{ A \in \mathfrak{g}_h^0 \mid A_{ij} = 0, \ \forall j \notin N(i) \cup \{i\}, \ \ A^\mathsf{T}\Theta + \Theta A \text{ is diagonal} \right\}$$

which is the nonholonomic constraint used in [3]. By Proposition 3.3, we have $\operatorname{Im} \mathsf{A} \simeq RT_{2r}(\mathcal{T}_h) = RT_0(\mathcal{T}_h)$ in the compressible case. This is reflected in (32): every nonzero off-diagonal entry of $A \in \operatorname{Im} \mathsf{A}$ corresponds to a degree of freedom $\int_e (u \cdot n) \mathrm{d}s$, $e \in \mathcal{E}_h^0$, for $RT_0(\mathcal{T}_h)$ (and every diagonal entry of A is a linear combination thereof).

4 The Lie algebra-to-vector fields map

In this section we define a Lie algebra-to-vector fields map that associates to a matrix $A \in \mathfrak{gl}(V_h^r)$ a vector field on Ω. Such a map is needed to define in a general way the semidiscrete Lagrangian associated to a given continuous Lagrangian.

Since any $A \in S_h^r$ is associated to a unique vector field $u \in RT_{2r}(\mathcal{T}_h)$, one could think that the correspondence $A \in S_r^h \to u \in RT_{2r}(\mathcal{T}_h)$ can be used as a Lie algebra-to-vector fields map. However, as explained in detail in Appendix B, the Lagrangian must be defined on a larger space than the constraint space S_h^r, namely, at least on $S_h^r + [S_h^r, S_h^r]$. This is why such a Lie algebra-to-vector fields map is needed.

Definition 4.1. For $r \geq 0$ an integer, we consider the Lie algebra-to-vector field map $\widehat{\ } : \mathfrak{gl}(V_h^r) \to [V_h^r]^n$ defined by

$$\widehat{A} := \sum_{k=1}^{n} A(I_h^r(x^k)) e_k, \tag{33}$$

where $I_h^r : L^2(\Omega) \to V_h^r$ is the L^2-orthogonal projector onto V_h^r, $x^k : \Omega \to \mathbb{R}$ are the coordinate maps, and e_k the canonical basis for $\mathbb{R}^n$. When $r \geq 1$, $I_h^r(x^k) = x^k$, whereas $I_h^0(x^k)$ equals the k^{th} component of the barycenter on each K. For $r \geq 1$ this map was considered in [18].

The idea leading to the definition (33) is the following. On one hand the component u^k of a general vector field $u = \sum_k u^k e_k$, can be understood as the derivative of the coordinate function x^k in the direction u, i.e. $u^k = \nabla_u x^k$. On the other hand, from the definition of the discrete diffeomorphism group, the linear map $f \mapsto Af$ for $f \in V_h^r$ is understood as a derivation, hence (33) is a natural candidate for a Lie algebra-to-vector field map. We shall study its properties below, after describing in more detail in the next lemma the expression (33) for $r = 0$.

Lemma 4.1. *For $r = 0$ and $A \in \mathfrak{g}_h^0 \subset \mathfrak{gl}(V_h^0)$, $\widehat{A}$ is the vector field constant on each simplex, given on simplex K_i by*

$$\widehat{A}|_{K_i} = \sum_j (b_j - b_i) A_{ij}, \tag{34}$$

where $b_i = \frac{1}{|K_i|}\int_{K_i} x \,\mathrm{d}x$ denotes the barycenter of K_i.

Proof. The L^2-projection of the coordinate function x^k onto V_h is given by

$$I_h^0(x^k) = \sum_j \psi_j \frac{1}{|K_j|}\int_{K_j} x^k = \sum_j \psi_j b_j^k,$$

where b_j^k denotes the k^{th} component of b_j. Hence,

$$\begin{aligned}\widehat{A} &= \sum_{k=1}^n (A(I_h x^k)) e_k = \sum_{k=1}^n \sum_j b_j^k e_k A\psi_j = \sum_j b_j A\psi_j \\ &= \sum_j b_j \sum_i A_{ij}\psi_i = \sum_i \psi_i \sum_j b_j A_{ij} = \sum_i \psi_i \sum_j (b_j - b_i) A_{ij},\end{aligned}$$

where the last equality follows from the fact that $\sum_j A_{ij} = 0$ for every i. □

Proposition 4.2. *For $u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)$, $p > 2$ and $r \geq 0$, we consider $A_u \in \mathfrak{gl}(V_h^r)$ defined in (23).*

- *If $r \geq 1$, then for all $u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)$, $p > 2$, we have*

  $$(\widehat{A_u})^k = I_h^r(u^k), \quad k = 1, ..., n.$$

  In particular, if u is such that $u|_K \in P_r(K)^n$ for all K, then $\widehat{A_u} = u$.

- *If $r = 0$, then*

  $$\widehat{A_u}|_K = \frac{1}{2|K|}\sum_{e \in K}\int_e u \cdot n_{e_-}(b_{e_+} - b_{e_-})\mathrm{d}s$$

  where n_{e_-} is the normal vector field pointing from K_- to K_+ and $b_{e_\pm}$ are the barycenters of $K_\pm$. In particular, if $u \in RT_0(\mathcal{T}_h)$, then

  $$\widehat{A_u}|_K = \frac{1}{2|K|}\sum_{e \in K} |e| u \cdot n_{e_-}(b_{e_+} - b_{e_-}).$$

  More particularly, if $u|_K \in P_0(K)^n$ for all K and the triangles are regular

  $$\widehat{A_u} = u.$$

As a consequence, we also note that for $r \geq 1$ and $u \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)$, $p > 2$:

$$\widehat{A_u} = u \quad \Leftrightarrow \quad u|_K \in P_r(K)^n, \ \forall K.$$

Proof. When $r \geq 1$, we have $I_h^r(x^k) = x^k$, hence $\widehat{A_u}(x) = \sum_{k=1}^n (A_u x^k)(x) e_k$. We compute $A_u x^k \in V_h^r$ as follows: for all $g \in V_h^r$, we have

$$\langle A_u x^k, g \rangle = \sum_{K \in \mathcal{T}_h} \int_K (\nabla x^k \cdot u) g \mathrm{d}x - \sum_{e \in \mathcal{E}_h^0} \int_e u \cdot [\![x^k]\!]_e \{g\}_e \mathrm{d}s = \int_\Omega u^k g \mathrm{d}x.$$

Since this is true for all $g \in V_h^r$ and since $A_u x^k$ must belong to V_h^r, we have

$$(\widehat{A_u})^k = A_u x^k = I_h^r(u^k),$$

as desired.

When $r = 0$, we have $I_h^0(f)|_{K_i} = \frac{1}{|K_i|} \int_{K_i} f(x) \mathrm{d}x$ hence $I_h^0(x^k)|_{K_i} = (b_i)^k$. We compute $A_u I_h^0(x^k) \in V_h^0$ as follows: for all $g \in V_h^0$, we have

$$\begin{aligned}
\langle A_u I_h^0(x^k), g \rangle &= \sum_{K_i \in \mathcal{T}_h} \int_{K_i} (\nabla (b_i)^k \cdot u) g \mathrm{d}x - \sum_{e \in \mathcal{E}_h^0} \int_e u \cdot \left(b_{e_+}^k n_{e_+} + b_{e_-}^k n_{e_-} \right) \{g\}_e \mathrm{d}s \\
&= 0 - \sum_{e \in \mathcal{E}_h^0} \int_e u \cdot \left(b_{e_+}^k n_{e_+} + b_{e_-}^k n_{e_-} \right) \frac{1}{2} (g_{e_+} + g_{e_-}) \mathrm{d}s \\
&= - \sum_{e \in \mathcal{E}_h^0} \int_e u \cdot \left(b_{e_+}^k n_{e_+} + b_{e_-}^k n_{e_-} \right) \mathrm{d}s \left(\frac{1}{2|K_{e_+}|} \int_{K_{e_+}} g \mathrm{d}x + \frac{1}{2|K_{e_-}|} \int_{K_{e_-}} g \mathrm{d}x \right) \\
&= - \sum_{K \in \mathcal{T}_h} \sum_{e \in K} \int_e u \cdot \left(b_{e_+}^k n_{e_+} + b_{e_-}^k n_{e_-} \right) \mathrm{d}s \frac{1}{2|K|} \int_K g \mathrm{d}x
\end{aligned}$$

hence we get

$$A_u I_h^0(x^k)|_K = \frac{1}{2|K|} \sum_{e \in K} \int_e u \cdot n_{e_-} \left(b_{e_+}^k - b_{e_-}^k \right) \mathrm{d}s$$

from which the result follows. This result can be also obtained by combining the results of Proposition 3.5 and Lemma 4.1.

In 2D, for the case of a regular triangle, we have $b_{e_+} - b_{e_-} = n_{e_-} \frac{2}{3} H$, where H is the height, and $|K| = \frac{1}{2} |e| H$ so we get

$$\widehat{A_u}|_K = \frac{1}{2|K|} \frac{2}{3} H \sum_{e \in K} |e| (u \cdot n_{e_-}) n_{e_-} = \frac{2}{3} \sum_{e \in K} (u \cdot n_{e_-}) n_{e_-} = u$$

if $u|_K \in P_0(K)^n$. Similar computations hold in 3D. □

Proposition 4.3. *For all $u, v \in H_0(\mathrm{div}, \Omega) \cap L^p(\Omega)$, $p > 2$, and $r \geq 1$, we have*

$$\langle \widehat{[A_u, A_v]}^k, g \rangle = \sum_{K \in \mathcal{T}_h} \int_K (\nabla \bar{v}^k \cdot u - \nabla \bar{u}^k \cdot v) g \mathrm{d}x - \sum_{e \in \mathcal{E}_h^0} \int_e \left(u \cdot n[\bar{v}^k] - v \cdot n[\bar{u}^k] \right) \{g\} \mathrm{d}s,$$

for $k=1,...,n$, for all $g\in V_h^r$, where $\bar u^k = I_h^r(u^k)\in V_h^r$ and $\bar v^k = I_h^r(v^k)\in V_h^r$. The convention is such that if n is pointing from K_- to K_+, then $[\bar v^k]=\bar v^k_- - \bar v^k_+$.

So, in particular if $u|_K, v|_K\in P_r(K)$, then

$$\langle \widehat{[A_u,A_v]}^k, g\rangle = \sum_{K\in\mathcal{T}_h}\int_K [u,v]^k g\mathrm{d}x - \sum_{e\in\mathcal{E}_h^0}\int_e \big(u\cdot n[v^k]-v\cdot n[u^k]\big)\{g\}\mathrm{d}s.$$

If $u,v,w\in H_0(\mathrm{div},\Omega)\cap L^p(\Omega)$, $p>2$ and $u|_K,v|_K,w|_K\in P_r(K)$, we have

$$\begin{aligned}\int_\Omega \widehat{[A_u,A_v]}\cdot\widehat{A_w}\,\mathrm{d}x &= \sum_{k=1}^n\langle \widehat{[A_u,A_v]}^k,\widehat{A_w}^k\rangle\\ &= \sum_{K\in\mathcal{T}_h}\int_K [u,v]\cdot w\,\mathrm{d}x - \sum_{e\in\mathcal{E}_h^0}\int_e (n\times\{w\})\cdot[u\times v]\mathrm{d}s.\end{aligned}$$

For $r=0$, and $u,v\in H_0(\mathrm{div},\Omega)\cap L^p(\Omega)$, $p>2$ such that $u|_K,v|_K\in P_0(K)$, then $\widehat{[A_u,A_v]}\in [V_h^0]^n$ is the vector field constant on each simplex K, given on K by

$$\widehat{[A_u,A_v]}|_K = \frac{1}{2|K|}\sum_{e\in K}|e|\Big(u\cdot n_{e_-}(c[v]_{e_+}-c[v]_{e_-}) - v\cdot n_{e_-}(c[u]_{e_+}-c[u]_{e_-})\Big),$$

where $c[u]\in[V_h^0]^n$ is the vector field constant on each simplex K, given on K by

$$c[u]_K = \frac{1}{2|K|}\sum_{e\in K}|e|u\cdot n_{e_-}(b_{e_+}-b_{e_-})$$

similarly for $c[v]\in[V_h^0]^n$.

Proof. We note that from Proposition 4.2,

$$\widehat{[A_u,A_v]}^k = A_u(A_v x^k) - A_v(A_u x^k) = A_u I_h^r(v^k) - A_v I_h^r(u^k) = A_u\bar v^k - A_v \bar u^k.$$

for all $u,v\in H_0(\mathrm{div},\Omega)\cap L^p(\Omega)$, $p>2$. Then, using (23), we have for all $g\in V_h^r$:

$$\langle A_u\bar v^k, g\rangle = \sum_{K\in\mathcal{T}_h}\int_K(\nabla\bar v^k\cdot u)g\mathrm{d}x - \sum_{e\in\mathcal{E}_h^0}\int_e u\cdot[\![\bar v^k]\!]\{g\}\mathrm{d}s$$

similarly for $\langle A_v\bar u^k, g\rangle$ from which we get the first formula.

The second formula follows when $u|_K,v|_K\in P_r(K)$ since in this case $u=\bar u$, $v=\bar v$.

For the third formula we choose $g=\widehat{A_w}^k=\bar w^k$ in the first formula and sum over $k=1,...,n$ to get

$$\begin{aligned}\int_\Omega \widehat{[A_u,A_v]}\cdot\widehat{A_w}\,\mathrm{d}x &= \sum_{k=1}^n\langle \widehat{[A_u,A_v]}^k,\widehat{A_w}^k\rangle\\ &= \sum_{K\in\mathcal{T}_h}\int_K(\nabla\bar v\cdot u - \nabla\bar u\cdot v)\cdot\bar w\,\mathrm{d}x\\ &\qquad - \sum_{e\in\mathcal{E}_h^0}\int_e\big(u\cdot n\,[\bar v]-v\cdot n\,[\bar u]\big)\cdot\{\bar w\}\mathrm{d}s.\end{aligned}$$

So far we only used $u, v, w \in H_0(\operatorname{div}, \Omega) \cap L^p(\Omega)$, $p > 2$. Now we assume further that $u|_K, v|_K, w|_K \in P_r(K)$ for all K, so we have $\bar{u} = u$, $\bar{v} = v$, $\bar{w} = w$, $u \cdot n = \{u\} \cdot n$, $v \cdot n = \{v\} \cdot n$, $[\bar{v}] \cdot n = [\bar{u}] \cdot n = 0$. Using some vector calculus identities for the last term, we get

$$\int_\Omega [\widehat{A_u, A_v}] \cdot \widehat{A_w} \, \mathrm{d}x = \sum_{K \in \mathcal{T}_h} \int_K [u, v] \cdot w \, \mathrm{d}x - \sum_{e \in \mathcal{E}_h^0} \int_e (n \times \{w\}) \cdot (\{u\} \times [v] + [u] \times \{v\}) \mathrm{d}s.$$

which yields the desired formula since $[u \times v] = \{u\} \times [v] + [u] \times \{v\}$. $\square$

For $r \geq 1$ the above result is the compressible version of Theorem 3.13 in [18].

5 Finite element variational integrator

In this section we derive the variational discretization for compressible fluids by using the setting developed so far. We focus on the case in which the Lagrangian depends only on the velocity and the mass density, since the extension to a dependence on the entropy density is straightforward, see §6 and Appendix A.

5.1 Semidiscrete Euler-Poincaré equations

Given a continuous Lagrangian $\ell(u, \rho)$ expressed in terms of the Eulerian velocity u and mass density ρ, the associated discrete Lagrangian $\ell_d : \mathfrak{g}_h^r \times V_h^r \to \mathbb{R}$ is defined with the help of the Lie algebra-to-vector fields map as

$$\ell_d(A, D) := \ell(\widehat{A}, D), \tag{35}$$

where $D \in V_h^r$ is the discrete density. Exactly as in the continuous case, the right action of G_h^r on discrete densities is defined by duality as

$$\langle D \cdot q, E \rangle = \langle D, qE \rangle, \quad \forall E \in V_h^r. \tag{36}$$

The corresponding action of $\mathfrak{g}_h^r$ on D is given by

$$\langle D \cdot B, E \rangle = \langle D, BE \rangle, \quad \forall E \in V_h^r. \tag{37}$$

The semidiscrete equations are derived by mimicking the variational formulation of the continuous equations, namely, by using the Euler-Poincaré principle applied to ℓ_d. As we have explained earlier, only the Lie algebra elements in $\operatorname{Im} \mathsf{A} = S_h^r$ actually represent a discretization of continuous vector fields. Following the approach initiated [19] this condition is included in the Euler-Poincaré principle by imposing S_h^r as a nonholonomic constraint, and hence applying the Euler-Poincaré-d'Alembert recalled in Appendix B. As we will see later, one needs to further restrict the constraint S_h^r to a subspace $\Delta_h^R \subset S_h^r$.

For a given constraint $\Delta_h^R \subset \mathfrak{g}_h^r$, a given Lagrangian ℓ_d, and a given duality pairing $\langle\langle K, A \rangle\rangle$ between elements $K \in (\mathfrak{g}_h^r)^*$ and $A \in \mathfrak{g}_h^r$, the Euler-Poincaré-d'Alembert principle seeks $A(t) \in \Delta_h^R$ and $D(t) \in V_h^r$ such that

$$\delta \int_0^T \ell_d(A, D) \mathrm{d}t = 0, \quad \text{for } \delta A = \partial_t B + [B, A] \text{ and } \delta D = -D \cdot B,$$

for all $B(t) \in \Delta_h^R$ with $B(0) = B(T) = 0$. The expressions for δA and δB are deduced from the relations $A(t) = \dot{q}(t) q(t)^{-1}$ and $D(t) = D_0 \cdot q(t)^{-1}$, with D_0 the initial value of the density, as in the continuous case.

The critical condition associated to this principle is

$$\left\langle\!\!\left\langle \partial_t \frac{\delta \ell_d}{\delta A}, B \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A}, [A, B] \right\rangle\!\!\right\rangle + \left\langle \frac{\delta \ell_d}{\delta D}, D \cdot B \right\rangle = 0, \quad \forall\, t \in (0, T), \quad \forall\, B \in \Delta_h^R, \tag{38}$$

or, equivalently,

$$\partial_t \frac{\delta \ell_d}{\delta A} + \mathrm{ad}_A^* \frac{\delta \ell_d}{\delta A} - \frac{\delta \ell_d}{\delta D} \diamond D \in (\Delta_h^R)^\circ, \quad \forall t \in (0, T). \tag{39}$$

The differential equation for D follows from differentiating $D(t) = D_0 \cdot q(t)^{-1}$ to obtain $\partial_t D = -D \cdot A$, or, equivalently,

$$\langle \partial_t D, E \rangle + \langle D, AE \rangle = 0, \quad \forall t \in (0, T), \quad \forall\, E \in V_h^r. \tag{40}$$

We refer to Appendix B for more details and the explanation of the notations. The extension of (38) and (39) to the case when the Lagrangian depends also on the entropy density is straightforward but important, see §6.

As explained in Appendix B, a sufficient condition for (38) to be a solvable system for T small enough is that the map

$$A \in \Delta_h^R \to \frac{\delta \ell_d}{\delta A}(A, D) \in (\mathfrak{g}_h^r)^* / (\Delta_h^R)^\circ \tag{41}$$

is a diffeomorphism for all $D \in V_h^r$ strictly positive.

5.2 The compressible fluid

We now focus on the compressible barotropic fluid, whose continuous Lagrangian is given in (14). Following (35), we have the discrete Lagrangian

$$\ell_d(A, D) := \ell(\widehat{A}, D) = \int_\Omega \left[\frac{1}{2} D |\widehat{A}|^2 - D e(D) \right] \mathrm{d}x. \tag{42}$$

In order to check condition (41), we shall compute the functional derivative $\frac{\delta \ell_d}{\delta A}$. We have

$$\left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A}, \delta A \right\rangle\!\!\right\rangle = \int_\Omega D \widehat{A} \cdot \widehat{\delta A} \mathrm{d}x = \int_\Omega I_h^r(D\widehat{A}) \cdot \widehat{\delta A} \mathrm{d}x = \left\langle\!\!\left\langle I_h^r(D\widehat{A})^\flat, \delta A \right\rangle\!\!\right\rangle$$

where we defined the linear map $\flat : ([V_h^r]^n)^* = [V_h^r]^n \to (\mathfrak{g}_h^r)^*$ as the dual map to $\widehat{\ }: \mathfrak{g}_h^r \to [V_h^r]^n$, namely

$$\langle\!\langle \alpha^\flat, A \rangle\!\rangle = \langle \alpha, \widehat{A} \rangle, \ \forall \alpha \in [V_h^r]^n, \ A \in \mathfrak{g}_h^r.$$

We thus get $\frac{\delta \ell_d}{\delta A} = I_h^r(D\widehat{A})^\flat$ and note that the choice $\Delta_h^R = S_h^r$ is not appropriate since the linear map $A \in S_h^r \mapsto I_h^r(D\widehat{A})^\flat \in (\mathfrak{g}_h^r)^*/(S_h^r)^\circ$ is not an isomorphism. We thus need to restrict the constraint S_h^r to a subspace $\Delta_h^R \subset S_h^r$ such that

$$A \in \Delta_h^R \mapsto I_h^r(D\widehat{A})^\flat \in (\mathfrak{g}_h^r)^*/(\Delta_h^R)^\circ \tag{43}$$

becomes an isomorphism, for all $D \in V_h^r$ strictly positive. We shall denote by R_h the subspace of $RT_{2r}(\mathcal{T}_h)$ corresponding to Δ_h^R via the isomorphism $u \in RT_{2r}(\mathcal{T}_h) \mapsto A_u \in S_h^r$ shown in Proposition

3.3. The diagram below illustrates the situation that we consider.

$$\begin{tikzcd}
H_0(\mathrm{div},\Omega) \arrow[r, "\mathsf{A}"] & S_h^r \arrow[r, hook] & \mathfrak{g}_h^r \arrow[r, "\widehat{\ }"] & {[V_h^r]^n} \\
RT_{2r}(\mathcal{T}_h) \arrow[u, hook] \arrow[ur, leftrightarrow] & \Delta_h^R \arrow[u, hook] & & \\
R_h \arrow[u, hook] \arrow[ur, leftrightarrow] & & &
\end{tikzcd}$$

The kernel of (43) is computed as

$$\begin{aligned}
\{A \in \Delta_h^R \mid I_h^r(D\widehat{A})^\flat \in (\Delta_h^R)^\circ\} &= \{A \in \Delta_h^R \mid \langle\!\langle I_h^r(D\widehat{A})^\flat, B\rangle\!\rangle = 0,\ \forall B \in \Delta_h^R\} \\
&= \{A \in \Delta_h^R \mid \langle I_h^r(D\widehat{A}), \widehat{B}\rangle = 0,\ \forall B \in \Delta_h^R\} \\
&= \{A_u \in \Delta_h^R \mid \langle I_h^r(DI_h^r(u)), I_h^r(v)\rangle = 0,\ \forall v \in R_h\} \\
&= \{A_u \in \Delta_h^R \mid \langle DI_h^r(u), I_h^r(v)\rangle = 0,\ \forall v \in R_h\}.
\end{aligned}$$

We note that since $A, B \in \Delta_h^R \subset S_h^r$, we have $A = A_u$ and $B = B_v$ for unique $u, v \in R_h \subset RT_{2r}(\mathcal{T}_h)$ by Proposition 3.3, so the kernel is isomorphic to the space

$$\{u \in R_h \mid \langle DI_h^r(u), I_h^r(v)\rangle = 0,\ \forall v \in R_h\}. \tag{44}$$

This space is zero if and only if R_h is a subspace of $[V_h^r]^n \cap H_0(\mathrm{div},\Omega) = BDM_r(\mathcal{T}_h)$, the Brezzi-Douglas-Marini finite element space of order r. Indeed, in this case the space (44) can be rewritten as

$$\{u \in R_h \mid \langle Du, v\rangle = 0,\ \forall v \in R_h\} = \{0\}$$

since D is strictly positive (it suffices to take $v = u$). Conversely, if there exists a nonzero $w \in R_h \setminus BDM_r(\mathcal{T}_h)$, then $u := w - I_h^r(w) \neq 0$ satisfies $I_h^r(u) = 0$, showing that (44) is nonzero.

Using the expressions of the functional derivatives

$$\frac{\delta \ell_d}{\delta A} = I_h^r(D\widehat{A})^\flat, \quad \frac{\delta \ell_d}{\delta D} = I_h^r\Big(\frac{1}{2}|\widehat{A}|^2 - e(D) - D\frac{\partial e}{\partial D}\Big),$$

of (42), the Euler-Poincaré equations (38) are equivalent to

$$\Big\langle \partial_t(D\widehat{A}), \widehat{B}\Big\rangle + \Big\langle D\widehat{A}, \widehat{[A,B]}\Big\rangle + \Big\langle I_h^r\Big(\frac{1}{2}|\widehat{A}|^2 - e(D) - D\frac{\partial e}{\partial D}\Big), D\cdot B\Big\rangle = 0, \quad \forall t \in (0,T), \quad \forall\, B \in \Delta_h^R. \tag{45}$$

To relate (45) and (40) to more traditional finite element notation, let us denote $\rho_h = D$, $u_h = -\widehat{A}$, $\sigma_h = E$, and $v_h = -\widehat{B}$. Then, using Proposition 4.3, the identities $\widehat{A_{u_h}} = -\widehat{A}$ and $\widehat{A_{v_h}} = -\widehat{B}$, and the definition (23) of A_u, we see that (45) and (40) are equivalent to seeking $u_h \in R_h$ and $\rho_h \in V_h^r$ such that

$$\langle \partial_t(\rho_h u_h), v_h\rangle + a_h(w_h, u_h, v_h) - b_h(v_h, f_h, \rho_h) = 0, \qquad \forall v_h \in R_h \tag{46}$$

$$\langle \partial_t \rho_h, \sigma_h\rangle - b_h(u_h, \sigma_h, \rho_h) = 0, \qquad \forall \sigma_h \in V_h^r, \tag{47}$$

where $w_h = I_h^r(\rho_h u_h)$, $f_h = I_h^r\left(\frac{1}{2}|u_h|^2 - e(\rho_h) - \rho_h \frac{\partial e}{\partial \rho_h}\right)$, and

$$a_h(w,u,v) = \sum_{K\in\mathcal{T}_h} \int_K w\cdot(v\cdot\nabla u - u\cdot\nabla v)\,dx + \sum_{e\in\mathcal{E}_h^0}\int_e (v\cdot n[u] - u\cdot n[v])\cdot\{w\}\,ds,$$

$$b_h(w,f,g) = \sum_{K\in\mathcal{T}_h}\int_K (w\cdot\nabla f)g\,dx - \sum_{e\in\mathcal{E}_h^0} w\cdot[\![f]\!]\{g\}\,ds.$$

Remark 5.1. The above calculations carry over also to the setting in which the density is taken to be an element of $V_h^s \subset V_h^r$, $s<r$. In this setting, (47) must hold for every $\sigma_h \in V_h^s$, the definition of f_h becomes $f_h = I_h^s\left(\frac{1}{2}|u_h|^2 - e(\rho_h) - \rho_h\frac{\partial e}{\partial \rho_h}\right)$, and the definition of w_h remains unchanged. By fixing s and R_h, we may then take r large enough so that $I_h^r(\rho_h u_h) = \rho_h u_h$.

Extension to rotating fluids. For the purpose of application in geophysical fluid dynamics, we consider the case of a rotating fluid with angular velocity ω in a gravitational field with potential $\phi(x)$. The equations of motion are obtained by taking the Lagrangian

$$\ell(u,\rho) = \int_\Omega \left[\frac{1}{2}\rho|u|^2 + \rho R\cdot u - \rho e(\rho) - \rho\phi\right]\mathrm{d}x,$$

where the vector field R is half the vector potential of ω, i.e. $2\omega = \operatorname{curl} R$. Application of the Euler-Poincaré principle (15) yields the balance of fluid momentum

$$\rho(\partial_t u + u\cdot\nabla u + 2\omega\times u) = -\rho\nabla\phi - \nabla p, \quad \text{with} \quad p = \rho^2\frac{\partial e}{\partial \rho}. \tag{48}$$

The discrete Lagrangian is defined exactly as in (35) and reads

$$\ell_d(A,D) := \ell(\widehat{A},D) = \int_\Omega \left[\frac{1}{2}D|\widehat{A}|^2 + D\widehat{A}\cdot R - De(D) - D\phi\right]\mathrm{d}x. \tag{49}$$

We get $\frac{\delta \ell_d}{\delta A} = I_h^r(D\widehat{A})^\flat + I_h(DR)^\flat$ and the same reasoning as before shows that the affine map

$$A\in\Delta_h^R \to \frac{\delta\ell_d}{\delta A} \in (\mathfrak{g}_h^r)^*/(\Delta_h^R)^\circ \tag{50}$$

is a diffeomorphism for all $D\in V_h^r$ strictly positive. The Euler-Poincaré equations (38) now yield

$$\left\langle \partial_t\big(D(\widehat{A}+R)\big), \widehat{B}\right\rangle + \left\langle D(\widehat{A}+R), \widehat{[A,B]}\right\rangle + \left\langle I_h^r\Big(\frac{1}{2}|\widehat{A}|^2 + \widehat{A}\cdot R - e(D) - D\frac{\partial e}{\partial D} - \phi\Big), D\cdot B\right\rangle = 0, \ \ \forall B\in\Delta_h^R, \tag{51}$$

which, in traditional finite element notations is

$$\langle \partial_t(\rho_h u_h + \rho_h R), v_h\rangle + a_h(w_h,u_h,v_h) - b_h(v_h,f_h,\rho_h) = 0, \qquad \forall v_h\in R_h, \tag{52}$$

$$\langle \partial_t\rho_h, \sigma_h\rangle - b_h(u_h,\sigma_h,\rho_h) = 0, \qquad \forall \sigma_h \in V_h^r, \tag{53}$$

where $w_h = I_h^r(\rho_h u_h + \rho_h R)$, $f_h = I_h^r\left(\frac{1}{2}|u_h|^2 + u_h\cdot R - e(\rho_h) - \rho_h\frac{\partial e}{\partial\rho_h} - \phi\right)$, and a_h, b_h defined as before.

Lowest-order setting. As a consequence of Remark 3.1, for $r=0$, the Euler-Poincaré equations (38) are identical to the discrete equations considered in [3] and, in the incompressible case, they coincide with those of [19, 11, 9]. The discrete Lagrangians used are however different. For instance, by using the result of Lemma 4.1, for $r=0$, the discrete Lagrangian (42) for $A \in S_h^0$ becomes

$$\ell(A,D) = \frac{1}{2}\sum_i |K_i| D_i \sum_{j,k\in N(i)} M_{jk}^{(i)} A_{ij}A_{ik} - \sum_i |K_i| D_i e(D_i), \tag{54}$$

where $M_{jk}^{(i)} = (b_j - b_i)\cdot(b_k - b_i)$. This is similar, but not identical, to the reduced Lagrangian used in, e.g., [19]. There, each $M^{(i)}$ is replaced by a diagonal matrix with diagonal entries $M_{jj}^{(i)} = 2|K_i||c_i - c_j|/|K_i \cap K_j|$, where c_i denotes the circumcenter of K_i.

Remark 5.2 (Function vs densities)**.** We have defined in §2 the discrete diffeomorphism group by considering its action on discrete functions. This corresponds to the discrete version of the right action $f \mapsto f\cdot\varphi = f\circ\varphi$ with associated Lie algebra action $f \mapsto f\cdot u = \nabla_u f$. We could have also defined the discrete diffeomorphism group by using a discrete version of the right action on densities $\rho\mapsto\rho\cdot\varphi = (\rho\circ\varphi)J\varphi$ whose associated Lie algebra action is $\rho\mapsto\rho\cdot u = \operatorname{div}(\rho u)$. If this option is chosen, then the natural corresponding discretization of vector fields, denoted $\widetilde{A}_u$, acts on discrete densities as

$$\langle \widetilde{A}_u\rho, f\rangle = \sum_{K\in\mathcal{T}_h}\int_K \operatorname{div}(\rho u) f \,\mathrm{d}x - \sum_{e\in\mathcal{E}_h^0}\int_e u\cdot [\![\rho]\!]\{f\}\,\mathrm{d}s$$

and one notes that

$$\big\langle \widetilde{A}_u\rho, f\big\rangle = -\big\langle \rho, A_u f\big\rangle.$$

In this setting, the Lie algebra-to-vector field map must be modified accordingly, i.e., it follows by applying the Lie algebra-to-vector fields map (33) to minus the adjoint of A. As a consequence, with this approach we get the same discrete Lagrangian and the same discrete equations of motion as previously.

Variational time discretization. The variational character of compressible fluid equations can be exploited also at the temporal level, by deriving the temporal scheme via a discretization in time of the Euler-Poincaré variational principle, in a similar way to what has been done in [11, 9] for incompressible fluid models. This discretization of the Euler-Poincaré equation follows the one presented in [6].

In this setting, the relations $A(t) = \dot{g}(t)g(t)^{-1}$ and $D(t) = D_0\cdot g(t)^{-1}$ are discretized as

$$A_k = \tau^{-1}(g_{k+1}g_k^{-1})/\Delta t \quad \text{and} \quad D_k = D_0\cdot g_k^{-1}, \tag{55}$$

where $\tau : \mathfrak{g}_h^r \to G_h^r$ is a local diffeomorphism from a neighborhood of $0\in\mathfrak{g}_h^r$ to a neighborhood of $e\in G_h^r$ with $\tau(0)=e$ and $\tau(A)^{-1} = \tau(-A)$. Given $A\in\mathfrak{g}_h^r$, we denote by $d\tau_A : \mathfrak{g}_h^r\to\mathfrak{g}_h^r$ the right trivialized tangent map defined as

$$d\tau_A(\delta A) := (\mathbf{D}\tau(A)\cdot\delta A)\,\tau(A)^{-1}, \quad \delta A\in\mathfrak{g}_h^r.$$

We denote by $d\tau_A^{-1}:\mathfrak{g}_h^r\to\mathfrak{g}_h^r$ its inverse and by $(d\tau_A^{-1})^* : (\mathfrak{g}_h^r)^* \to (\mathfrak{g}_h^r)^*$ the dual map.

The discrete Euler-Poincaré-d'Alembert variational principle reads

$$\delta\sum_{k=0}^{K-1}\ell_d(A_k, D_k)\Delta t = 0,$$

for variations

$$\delta A_k = \frac{1}{\Delta t} d\tau^{-1}_{\Delta t A_k}(B_{k+1}) - \frac{1}{\Delta t} d\tau^{-1}_{-\Delta t A_k}(B_k), \qquad \delta D_k = -D_k \cdot B_k$$

where $B_k \in \Delta_h^R$ vanishes at the extremities. These variations are obtained by taking the variations of the relations (55) and defining $B_k = \delta g_k g_k^{-1}$. It yields

$$\frac{1}{\Delta t}\left\langle\!\!\left\langle \left(d\tau^{-1}_{\Delta t A_{k-1}}\right)^* \frac{\delta \ell_d}{\delta A_{k-1}} - \left(d\tau^{-1}_{-\Delta t A_k}\right)^* \frac{\delta \ell_d}{\delta A_k}, B_k \right\rangle\!\!\right\rangle - \left\langle \frac{\delta \ell_d}{\delta D_k}, D_k \cdot B_k \right\rangle = 0, \qquad \forall\, B_k \in \Delta_h^R.$$

From $D_k = D_0 \cdot g_k^{-1}$, one gets

$$D_{k+1} = D_k \cdot \tau(-\Delta t A_k). \tag{56}$$

Several choices are possible for the local diffeomorphism τ, see, e.g., [6]. One option is the Cayley transform

$$\tau(A) = \left(I - \frac{A}{2}\right)^{-1}\left(I + \frac{A}{2}\right).$$

We have $\tau(0) = I$ and since $\mathbf{D}\tau(0)\delta A = \delta A$, it is a local diffeomorphism. We also note that $A\mathbf{1} = 0$ implies $\tau(A)\mathbf{1} = \mathbf{1}$ in a suitably small neighborhood of 0. We have

$$d\tau_A(\delta A) = \left(I - \frac{A}{2}\right)^{-1} \delta A \left(I + \frac{A}{2}\right)^{-1}, \quad d\tau_A^{-1}(B) = B + \frac{1}{2}[B, A] - \frac{1}{4}ABA,$$

so the discrete Euler-Poincaré equations read

$$\begin{aligned} &\left\langle\!\!\left\langle \frac{1}{\Delta t}\left(\frac{\delta \ell_d}{\delta A_k} - \frac{\delta \ell_d}{\delta A_{k-1}}\right), B_k \right\rangle\!\!\right\rangle + \frac{1}{2}\left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A_k}, [A_k, B_k] - \frac{\Delta t}{2} A_k B_k A_k \right\rangle\!\!\right\rangle \\ &+ \frac{1}{2}\left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A_{k-1}}, [A_{k-1}, B_k] + \frac{\Delta t}{2} A_{k-1} B_k A_{k-1} \right\rangle\!\!\right\rangle + \left\langle \frac{\delta \ell_d}{\delta D_k}, D_k \cdot B_k \right\rangle = 0, \quad \forall\, B_k \in \Delta_h^R. \end{aligned} \tag{57}$$

This is the discrete time version of (38). The discrete time version of (39) can be similarly written. With this choice of τ, the evolution D_k is obtained from (56), which is equivalent to

$$D_k \cdot (I + \frac{\Delta t}{2} A_{k-1}) = D_{k-1} \cdot (I - \frac{\Delta t}{2} A_{k-1}).$$

Recalling (37), we get

$$\left\langle \frac{D_k - D_{k-1}}{\Delta t}, E_k \right\rangle + \left\langle \frac{D_{k-1} + D_k}{2}, A_{k-1} E_k \right\rangle = 0, \quad \forall E_k \in V_h^r. \tag{58}$$

Energy preserving time discretization. For Lagrangians of the form (49), it is possible to construct a time discretization that exactly preserves the energy $\int_\Omega [\frac{1}{2} D|\widehat{A}|^2 + De(D) + D\phi]\mathrm{d}x$. Note that the contribution of the rotation does not appear in the expression of the total energy. To do this, let us define

$$F_{k-1/2} = \frac{1}{2}\widehat{A_{k-1}} \cdot \widehat{A_k} + \widehat{A_{k-1/2}} \cdot R - f(D_{k-1}, D_k) - \phi, \tag{59}$$

where

$$f(x, y) = \frac{ye(y) - xe(x)}{y - x}$$

and $A_{k-1/2} = \frac{1}{2}(A_{k-1} + A_k)$. Also let $D_{k-1/2} = \frac{1}{2}(D_{k-1} + D_k)$. The energy-preserving scheme reads

$$\begin{aligned} &\left\langle\!\!\left\langle \frac{1}{\Delta t}\left(\frac{\delta \ell_d}{\delta A_k} - \frac{\delta \ell_d}{\delta A_{k-1}}\right), B_k \right\rangle\!\!\right\rangle \\ &+\frac{1}{2}\left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A_{k-1}} + \frac{\delta \ell_d}{\delta A_k}, [A_{k-1/2}, B_k] \right\rangle\!\!\right\rangle + \langle F_{k-1/2}, D_{k-1/2} \cdot B_k \rangle = 0, \quad \forall\, B_k \in \Delta_h^R, \end{aligned} \tag{60}$$

$$\left\langle \frac{D_k - D_{k-1}}{\Delta t}, E_k \right\rangle + \langle D_{k-1/2} \cdot A_{k-1/2}, E_k \rangle = 0, \quad \forall E_k \in V_h^r. \tag{61}$$

Proposition 5.1. *The solution of (60-61) satisfies*

$$\int_\Omega \left[\frac{1}{2} D_k |\widehat{A_k}|^2 + D_k e(D_k) + D_k \phi\right] \mathrm{d}x = \int_\Omega \left[\frac{1}{2} D_{k-1} |\widehat{A_{k-1}}|^2 + D_{k-1} e(D_{k-1}) + D_{k-1}\phi\right] \mathrm{d}x. \tag{62}$$

Proof. Taking $B_k = A_{k-1/2}$ in (60) gives

$$\left\langle\!\!\left\langle \frac{1}{\Delta t}\left(\frac{\delta \ell_d}{\delta A_k} - \frac{\delta \ell_d}{\delta A_{k-1}}\right), A_{k-1/2} \right\rangle\!\!\right\rangle + \langle F_{k-1/2}, D_{k-1/2} \cdot A_{k-1/2} \rangle = 0.$$

Using the density equation (61) and the definition (49) of ℓ_d, we can rewrite this as

$$\left\langle \frac{1}{\Delta t}\left(D_k(\widehat{A_k} + R) - D_{k-1}(\widehat{A_{k-1}} + R)\right), \widehat{A_{k-1/2}} \right\rangle - \left\langle \frac{D_k - D_{k-1}}{\Delta t}, F_{k-1/2} \right\rangle = 0. \tag{63}$$

After rearrangement, the first term can be expressed as

$$\begin{aligned} &\left\langle \frac{1}{\Delta t}\left(D_k(\widehat{A_k} + R) - D_{k-1}(\widehat{A_{k-1}} + R)\right), \widehat{A_{k-1/2}} \right\rangle \\ &= \frac{1}{2\Delta t}\left(\langle D_k \widehat{A_k}, \widehat{A_k}\rangle - \langle D_{k-1}\widehat{A_{k-1}}, \widehat{A_{k-1}}\rangle\right) + \left\langle \frac{D_k - D_{k-1}}{\Delta t}, \widehat{A_{k-1/2}} \cdot R + \frac{1}{2}\widehat{A_{k-1}} \cdot \widehat{A_k} \right\rangle. \end{aligned}$$

Inserting this and the definition of $F_{k-1/2}$ into (63) gives

$$\frac{1}{2\Delta t}\left(\langle D_k \widehat{A_k}, \widehat{A_k}\rangle - \langle D_{k-1}\widehat{A_{k-1}}, \widehat{A_{k-1}}\rangle\right) + \left\langle \frac{D_k - D_{k-1}}{\Delta t}, f(D_{k-1}, D_k) + \phi \right\rangle = 0.$$

Finally, the definition of f yields

$$\frac{1}{2\Delta t}\left(\langle D_k \widehat{A_k}, \widehat{A_k}\rangle - \langle D_{k-1}\widehat{A_{k-1}}, \widehat{A_{k-1}}\rangle\right) + \left\langle \frac{D_k e(D_k) - D_{k-1} e(D_{k-1}) + D_k\phi - D_{k-1}\phi}{\Delta t}, 1 \right\rangle = 0,$$

which is equivalent to (62). □

Note that the definition of $F_{k-1/2}$ in (59) can be rewritten in terms of ℓ_d as

$$\ell_d(A_k, D_k) - \ell_d(A_{k-1}, D_{k-1}) = \frac{1}{2}\left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A_{k-1}} + \frac{\delta \ell_d}{\delta A_k}, A_k - A_{k-1} \right\rangle\!\!\right\rangle + \langle F_{k-1/2}, D_k - D_{k-1} \rangle, \tag{64}$$

This is reminiscent of a discrete gradient method [14, p. 174], with $F_{k-1/2}$ playing the role of the discrete version of $\frac{\delta \ell}{\delta D}$.

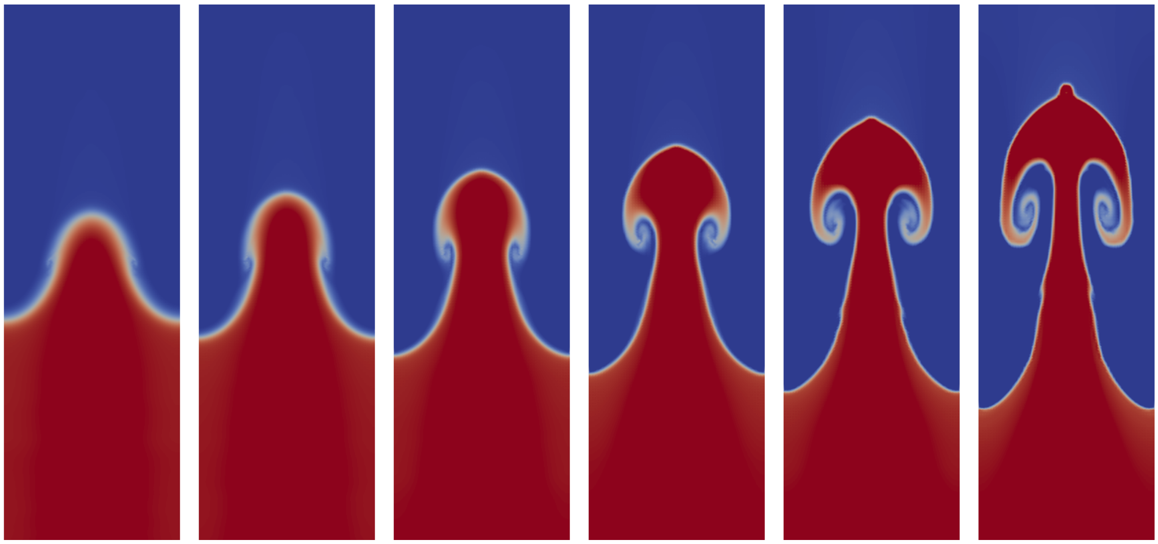

Figure 1: Contours of the mass density at $t = 1.0, 1.2, 1.4, 1.6, 1.8, 2.0$ in the Rayleigh-Taylor instability simulation with the energy-preserving time discretization (60-61).

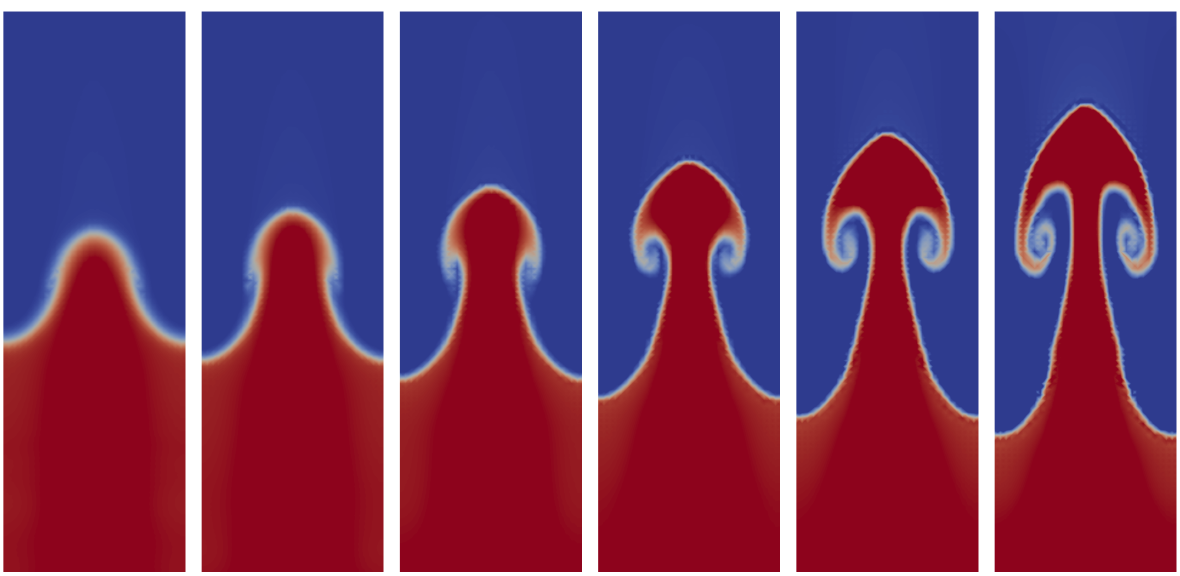

Figure 2: Contours of the mass density at $t = 1.0, 1.2, 1.4, 1.6, 1.8, 2.0$ in the Rayleigh-Taylor instability simulation with the variational time discretization (57-58).

r	h^{-1}	$\|u_h - u\|_{L^2(\Omega)}$	Rate	$\|\rho_h - \rho\|_{L^2(\Omega)}$	Rate
0	1	$3.58 \cdot 10^{-1}$		$2.10 \cdot 10^{-1}$	
	2	$1.84 \cdot 10^{-1}$	0.96	$1.17 \cdot 10^{-1}$	0.85
	4	$9.31 \cdot 10^{-2}$	0.99	$5.58 \cdot 10^{-2}$	1.06
	8	$4.64 \cdot 10^{-2}$	1.00	$2.74 \cdot 10^{-2}$	1.03
1	1	$1.43 \cdot 10^{-1}$		$1.00 \cdot 10^{-1}$	
	2	$4.36 \cdot 10^{-2}$	1.71	$2.43 \cdot 10^{-2}$	2.05
	4	$1.37 \cdot 10^{-2}$	1.68	$6.85 \cdot 10^{-3}$	1.83
	8	$4.40 \cdot 10^{-3}$	1.63	$1.74 \cdot 10^{-3}$	1.97
2	1	$2.78 \cdot 10^{-2}$		$1.83 \cdot 10^{-2}$	
	2	$7.80 \cdot 10^{-3}$	1.83	$4.61 \cdot 10^{-3}$	1.99
	4	$1.81 \cdot 10^{-3}$	2.11	$6.35 \cdot 10^{-4}$	2.86
	8	$4.50 \cdot 10^{-4}$	2.00	$1.15 \cdot 10^{-4}$	2.46

Table 1: L^2-errors in the velocity and density at time $T = 0.5$ obtained with the energy-preserving time discretization (60-61).

Δt^{-1}	$\|u_h - u\|_{L^2(\Omega)}$	Rate	$\|\rho_h - \rho\|_{L^2(\Omega)}$	Rate
2	$4.93 \cdot 10^{-2}$		$9.95 \cdot 10^{-2}$	
4	$1.68 \cdot 10^{-2}$	1.55	$3.12 \cdot 10^{-2}$	1.67
8	$5.03 \cdot 10^{-3}$	1.74	$8.92 \cdot 10^{-3}$	1.81
16	$1.44 \cdot 10^{-3}$	1.80	$2.43 \cdot 10^{-3}$	1.88

Table 2: Convergence with respect to Δt of the L^2-errors in the velocity and density at time $T = 0.5$ obtained with the energy-preserving time discretization (60-61).

6 Numerical tests

Convergence. To test our numerical method, we used (52-53) to simulate a rotating fluid with angular velocity $\omega = 1$ (i.e. $R = (-y, x)$) and internal energy $e(\rho) = \frac{1}{2}\rho$ in the absence of a gravitational field. This choice of the function $e(\rho)$ corresponds to the case of the rotating shallow water equations, for which ρ is interpreted as the fluid depth. We initialized $u(x, y, 0) = (0, 0)$ and $\rho(x, y, 0) = 2 + \sin(\pi x/2)\sin(\pi y/2)$ on $\Omega = (-1, 1) \times (-1, 1)$ and numerically integrated (52-53) using the energy-preserving time discretization (60-61) with $\Delta t = 0.00625$. We used the finite element spaces $R_h = RT_r(\mathcal{T}_h)$ and V_h^r with $r = 0, 1, 2$ on a uniform triangulation $\mathcal{T}_h$ of Ω with maximum element diameter $h = 2^{-j}$, $j = 0, 1, 2, 3$. We computed the L^2-errors in the velocity and density at time $T = 0.5$ by comparing with an "exact solution" obtained with $h = 2^{-5}$, $r = 2$. The results in Table 1 indicate that the method's convergence order is optimal (order $r+1$) when $r = 0$ and suboptimal when $r > 0$, but still grows with r.

We also repeated the above experiment with varying values of Δt and with fixed values of $h = 2^{-4}$ and $r = 2$. The results in Table 2 indicate that the method is second-order accurate with respect to Δt.

Lastly, we repeated both of the above experiments with the variational time discretization (57-58) in place of the energy-preserving time discretization (60-61). Following [19, 11, 9, 3], we discarded the terms $A_k B_k A_k$ and $A_{k-1} B_k A_{k-1}$ from (57) in our implementation. The results for the h-refinement experiment (not shown) were nearly indistinguishable from Table 1 once a sufficiently small time step ($\Delta t \approx 4 \times 10^{-4}$) was identified. The results for the Δt-refinement experiment, shown in Table 3, indicate that the variational time discretization (57-58) is first-order accurate with respect to Δt. The numbers labelled "Rate" in Table 3 are obtained from the errors ε_j and time steps Δt_j via $\log(\Delta t_{j-1}/\Delta t_j)/\log(\varepsilon_{j-1}/\varepsilon_j)$, $j = 2, 3, 4$. Note that we used smaller time

Δt^{-1}	$\|u_h - u\|_{L^2(\Omega)}$	Rate	$\|\rho_h - \rho\|_{L^2(\Omega)}$	Rate
450	$1.08 \cdot 10^{-3}$		$3.22 \cdot 10^{-3}$	
470	$1.03 \cdot 10^{-3}$	1.02	$3.07 \cdot 10^{-3}$	1.03
500	$9.74 \cdot 10^{-4}$	0.98	$2.89 \cdot 10^{-3}$	1.02
540	$9.04 \cdot 10^{-4}$	0.97	$2.67 \cdot 10^{-3}$	1.02

Table 3: Convergence with respect to Δt of the L^2-errors in the velocity and density at time $T = 0.5$ obtained with the variational time discretization (57-58).

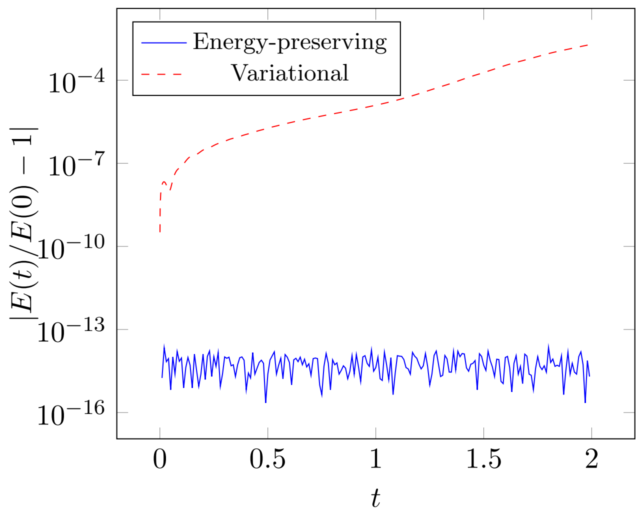

Figure 3: Relative errors in the energy $E(t) = \int_\Omega \left(\frac{1}{2}\rho|u|^2 + \rho e(\rho) + \rho\phi\right) \mathrm{d}x$ during the Rayleigh-Taylor instability simulation.

steps in Table 3 than in Table 2 for stability reasons; we observed numerically that the variational time discretization (57-58) is only conditionally stable, with a time step restriction of the form $\Delta t \leq Ch$. We believe the conditional stability of (57-58) is tied to (58), which is linearly implicit and decoupled from (57). Replacing A_{k-1} with $A_{k-1/2}$ in (58) appears to eliminate the instability.

Rayleigh-Taylor instability. Next, we simulated a Rayleigh-Taylor instability. For this test, we considered a fully (or baroclinic) compressible fluid, whose energy depends on both the mass density ρ and the entropy density s, both of which are advected parameters. The setting is the same as above, but with a Lagrangian

$$\ell(u, \rho, s) = \int_\Omega \left[\frac{1}{2}\rho|u|^2 - \rho e(\rho, \eta) - \rho\phi\right] \mathrm{d}x, \tag{65}$$

where $\eta = \frac{s}{\rho}$ is the specific entropy. In terms of the discrete velocity $A \in \Delta_h^R$, discrete mass density $D \in V_h^r$, and discrete entropy density $S \in V_h^r$, the spatially discrete Euler-Poincaré equations for this Lagrangian read

$$\left\langle\!\!\left\langle \partial_t \frac{\delta \ell_d}{\delta A}, B \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A}, [A, B] \right\rangle\!\!\right\rangle + \left\langle \frac{\delta \ell_d}{\delta D}, D \cdot B \right\rangle + \left\langle \frac{\delta \ell_d}{\delta S}, S \cdot B \right\rangle = 0, \quad \forall\, t \in (0, T), \quad \forall\, B \in \Delta_h^R, \tag{66}$$

or, equivalently,

$$\partial_t \frac{\delta \ell_d}{\delta A} + \mathrm{ad}^*_A \frac{\delta \ell_d}{\delta A} - \frac{\delta \ell_d}{\delta D} \diamond D - \frac{\delta \ell_d}{\delta S} \diamond S \in (\Delta_h^R)^\circ, \quad \forall t \in (0, T), \tag{67}$$

together with

$$\langle \partial_t D, E\rangle + \langle D, AE\rangle = 0, \quad \text{and} \quad \langle \partial_t S, H\rangle + \langle S, AH\rangle = 0, \quad \forall t \in (0,T), \quad \forall\, E, H \in V_h^r. \tag{68}$$

In analogy with (60-61), an energy-preserving time discretization is given by

$$\left\langle\!\!\left\langle \frac{1}{\Delta t}\left(\frac{\delta \ell_d}{\delta A_k} - \frac{\delta \ell_d}{\delta A_{k-1}}\right), B_k \right\rangle\!\!\right\rangle + \frac{1}{2}\left\langle\!\!\left\langle \frac{\delta \ell_d}{\delta A_{k-1}} + \frac{\delta \ell_d}{\delta A_k}, [A_{k-1/2}, B_k] \right\rangle\!\!\right\rangle$$
$$+\langle F_{k-1/2}, D_{k-1/2}\cdot B_k\rangle + \langle G_{k-1/2}, S_{k-1/2}\cdot B_k\rangle = 0, \quad \forall\, B_k \in \Delta_h^R, \tag{69}$$
$$\left\langle \frac{D_k - D_{k-1}}{\Delta t}, E_k \right\rangle + \langle D_{k-1/2}\cdot A_{k-1/2}, E_k\rangle = 0, \quad \forall E_k \in V_h^r, \tag{70}$$
$$\left\langle \frac{S_k - S_{k-1}}{\Delta t}, H_k \right\rangle + \langle S_{k-1/2}\cdot A_{k-1/2}, H_k\rangle = 0, \quad \forall H_k \in V_h^r, \tag{71}$$

where $A_{k-1/2} = \frac{1}{2}(A_{k-1} + A_k)$, $D_{k-1/2} = \frac{1}{2}(D_{k-1} + D_k)$, $S_{k-1/2} = \frac{1}{2}(S_{k-1} + S_k)$,

$$F_{k-1/2} = \frac{1}{2}\widehat{A_{k-1}}\cdot\widehat{A_k} - \frac{1}{2}\left(f(D_{k-1}, D_k, S_{k-1}) + f(D_{k-1}, D_k, S_k)\right) - \phi, \tag{72}$$
$$G_{k-1/2} = -\frac{1}{2}\left(g(S_{k-1}, S_k, D_{k-1}) + g(S_{k-1}, S_k, D_k)\right), \tag{73}$$

and

$$f(D, D', S) = \frac{D'e(D', S/D') - De(D, S/D)}{D' - D},$$
$$g(S, S', D) = \frac{De(D, S'/D) - De(D, S/D)}{S' - S}.$$

We took e equal to the internal energy for a perfect gas,

$$e(\rho, \eta) = Ke^{\eta/C_v}\rho^{\gamma - 1},$$

where $\gamma = 5/3$ and $K = C_v = 1$, and we used a gravitational potential $\phi = -y$, which corresponds to an upward gravitational force. We initialized

$$\rho(x, y, 0) = 1.5 - 0.5\tanh\left(\frac{y - 0.5}{0.02}\right),$$
$$u(x, y, 0) = \left(0, -0.025\sqrt{\frac{\gamma p(x,y)}{\rho(x,y,0)}}\cos(8\pi x)\exp\left(-\frac{(y-0.5)^2}{0.09}\right)\right),$$
$$s(x, y, 0) = C_v\rho(x, y, 0)\log\left(\frac{p(x,y)}{(\gamma - 1)K\rho(x,y,0)^\gamma}\right),$$

where

$$p(x, y) = 1.5y + 1.25 + (0.25 - 0.5y)\tanh\left(\frac{y - 0.5}{0.02}\right).$$

We implemented (69-71) with $\Delta t = 0.01$ and with the finite element spaces $R_h = RT_0(\mathcal{T}_h)$ and V_h^1 on a uniform triangulation $\mathcal{T}_h$ of $\Omega = (0, 1/4)\times(0, 1)$ with maximum element diameter $h = 2^{-8}$. We incorporated upwinding into (69-71) using the strategy detailed in [12], which retains the scheme's energy-preserving property. Plots of the computed mass density at various times t are shown in Fig. 1. Fig. 3 confirms that energy was preserved exactly up to roundoff errors.

We repeated this simulation with the variational time discretization (57-58) (modified to include the entropy density). To ensure stability, we used a smaller time step $\Delta t = 0.00025$ and incorporated upwinding as above. We used the same finite element spaces as above but with $h = 2^{-7}$. Energy decayed by less than 2×10^{-3} of its initial value; see Fig. 3. Our experiments suggest that this (rather insignificant) energy drift is attributable to upwinding, which cannot be abandoned without sacrificing stability in this example. Plots of the computed mass density are shown in Fig. 2.

A Euler-Poincaré variational principle

In this Appendix we first recall the Euler-Poincaré principle for invariant Euler-Lagrange systems on Lie groups. This general setting underlies the Lie group description of incompressible flows recalled in §2.1 due to [2], in which case the Lie group is $G = \mathrm{Diff}_{\mathrm{vol}}(\Omega)$. It also underlies the semidiscrete setting, in which case the Lie group is $G = G_h$. In this situation, however, a nonholonomic constraint needs to be considered, see Appendix B. Then, we describe the extension of this setting that is needed to formulate the variational formulation of compressible flow and its discretization.

A.1 Euler-Poincaré variational principle for incompressible flows

Let G be a Lie group and let $L : TG \to \mathbb{R}$ be a Lagrangian defined on the tangent bundle TG of G. The associated equations of evolution, given by the Euler-Lagrange equations, arise as the critical curve condition for the Hamilton principle

$$\delta \int_0^T L(g(t), \dot{g}(t))\mathrm{d}t = 0, \tag{74}$$

for arbitrary variations δg with $\delta g(0) = \delta g(T) = 0$.

If we assume that L is G-invariant, i.e., $L(gh, \dot{g}h) = L(g, \dot{g})$, for all $h \in G$, then L induces a function $\ell : \mathfrak{g} \to \mathbb{R}$ on the Lie algebra $\mathfrak{g}$ of G, defined by $\ell(u) = L(g, \dot{g})$, with $u = \dot{g}g^{-1} \in \mathfrak{g}$. In this case the equations of motion can be expressed exclusively in terms of u and ℓ and are obtained by rewriting the variational principle (74) in terms of ℓ and $u(t)$. One gets

$$\delta \int_0^T \ell(u(t))\mathrm{d}t = 0, \quad \text{for } \delta u = \partial_t v + [v, u], \tag{75}$$

where $v(t) \in \mathfrak{g}$ is an arbitrary curve with $v(0) = v(T) = 0$. The form of the variation δu in (75) is obtained by a direct computation using $u = \dot{g}g^{-1}$ and defining $v = \delta g g^{-1}$.

In order to formulate the equations associated to (75) one needs to select an appropriate space in nondegenerate duality with $\mathfrak{g}$ denoted $\mathfrak{g}^*$ (the usual dual space in finite dimensions). We shall denote by $\langle\!\langle\, , \rangle\!\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the associated nondegenerate duality pairing. From (75) one directly obtains the equation

$$\left\langle\!\!\left\langle \partial_t \frac{\delta \ell}{\delta u}, v \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{\delta \ell}{\delta u}, [u, v] \right\rangle\!\!\right\rangle = 0, \quad \forall\, v \in \mathfrak{g}. \tag{76}$$

In (76), the functional derivative $\frac{\delta \ell}{\delta u} \in \mathfrak{g}^*$ of ℓ is defined in terms of the duality pairing as

$$\left\langle\!\!\left\langle \frac{\delta \ell}{\delta u}, \delta u \right\rangle\!\!\right\rangle = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \ell(u + \epsilon \delta u).$$

In finite dimensions, and under appropriate choices for the functional spaces in infinite dimensions, (76) is equivalent to the Euler-Poincaré equation

$$\partial_t \frac{\delta \ell}{\delta u} + \operatorname{ad}^*_u \frac{\delta \ell}{\delta u} = 0,$$

where the coadjoint operator $\operatorname{ad}^*_u : \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by $\langle\!\langle \operatorname{ad}^*_u m, v \rangle\!\rangle = \langle\!\langle m, [u, v] \rangle\!\rangle$.

For incompressible flows, without describing the functional analytic setting for simplicity, we have $G = \operatorname{Diff}_{\mathrm{vol}}(\Omega)$ and $\mathfrak{g} = \mathfrak{X}_{\mathrm{vol}}(\Omega)$ the Lie algebra of divergence free vector fields parallel to the boundary. We can choose $\mathfrak{g}^* = \mathfrak{X}_{\mathrm{vol}}(\Omega)$ with duality pairing $\langle\!\langle\, , \rangle\!\rangle$ given by the L^2 inner product. A direct computation gives the coadjoint operator $\operatorname{ad}^*_u m = \mathbf{P}(u \cdot \nabla m + \nabla u^{\mathsf{T}} m)$, where $\mathbf{P}$ is the Leray-Hodge projector onto $\mathfrak{X}_{\mathrm{vol}}(\Omega)$. One directly checks that in this case (75) yields the Euler equations (3) for incompressible flows.

A.2 Euler-Poincaré variational principle for compressible flows

The general setting underlying the variational formulation for compressible fluids starts exactly as before, namely, a system whose evolution is given by the Euler-Lagrange equations for a Lagrangian defined on the tangent bundle of a Lie group G. The main difference is that the Lagrangian depends parametrically on some element $a_0 \in V$ of a vector space (the reference mass density ϱ_0 in the case of the barotropic compressible fluid, the reference mass and entropy densities ϱ_0 and S_0 for the general compressible fluid) on which G acts by representation, and, in addition, L is invariant only under the subgroup of G that keeps a_0 fixed. If we denote by $L(g, \dot{g}, a_0)$ this Lagrangian and by $a \in V \mapsto a \cdot g \in V$ the representation of G on V, the reduced Lagrangian is defined by $\ell(u, a) = L(g, \dot{g}, a_0)$, where $u = \dot{g} g^{-1}$, $a = a_0 \cdot g^{-1}$.

The Hamilton principle now yields the variational formulation

$$\delta \int_0^T \ell(u(t), a(t)) \mathrm{d}t = 0, \quad \text{for } \delta u = \partial_t v + [v, u] \text{ and } \delta a = -a \cdot v, \tag{77}$$

where $v(t) \in \mathfrak{g}$ is an arbitrary curve with $v(0) = v(T) = 0$. The form of the variation δu in (77) is the same as before, while the expression for δa is obtained from the relation $a = a_0 \cdot g^{-1}$.

From (77) and with respect to the choice of a spaces $\mathfrak{g}^*$ and V^* in nondegenerate duality with $\mathfrak{g}$ and V, with duality pairings $\langle\!\langle\, , \rangle\!\rangle$ and $\langle\, , \rangle_V$, one directly obtains the equations

$$\left\langle\!\!\left\langle \partial_t \frac{\delta \ell}{\delta u}, v \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{\delta \ell}{\delta u}, [u, v] \right\rangle\!\!\right\rangle + \left\langle \frac{\delta \ell}{\delta a}, a \cdot v \right\rangle_V = 0, \quad \forall\, v \in \mathfrak{g}. \tag{78}$$

The continuity equation

$$\partial_t a + a \cdot u = 0$$

arises from the definition $a(t) = a_0 \cdot g(t)^{-1}$. In a similar way with above, (78) now yields the Euler-Poincaré equations

$$\partial_t \frac{\delta \ell}{\delta u} + \operatorname{ad}^*_u \frac{\delta \ell}{\delta u} = \frac{\delta \ell}{\delta a} \diamond a, \tag{79}$$

where $\frac{\delta \ell}{\delta a} \diamond a \in \mathfrak{g}^*$ is defined by $\langle\!\langle \frac{\delta \ell}{\delta a} \diamond a, v \rangle\!\rangle = -\left\langle \frac{\delta \ell}{\delta a}, a \cdot v \right\rangle_V$, for all $v \in \mathfrak{g}$. We refer to [15] for a detailed exposition.

For the compressible fluid, in the continuous case we have $G = \operatorname{Diff}(\Omega)$ and $\mathfrak{g} = \mathfrak{X}(\Omega)$ the Lie algebra of vector fields on Ω with vanishing normal component to the boundary. We choose to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the L^2 duality pairing. Consider the Lagrangian (65) of the general compressible fluid. Using the expressions $\mathrm{ad}^*_u m = u \cdot \nabla m + \nabla u^\mathsf{T} m + m \operatorname{div} u$, $\frac{\delta \ell}{\delta u} = \rho u$, $\frac{\delta \ell}{\delta \rho} = \frac{1}{2}|u|^2 - e(\rho) - \rho \frac{\partial e}{\partial \rho} + \eta \frac{\partial e}{\partial \eta} - \phi$, $\frac{\delta \ell}{\delta s} = -\frac{\partial e}{\partial \eta}$, $\frac{\delta \ell}{\delta \rho} \diamond \rho = \rho \nabla \frac{\delta \ell}{\delta \rho}$, and $\frac{\delta \ell}{\delta s} \diamond s = s \nabla \frac{\delta \ell}{\delta s}$, one directly obtains

$$\rho(\partial_t u + u \cdot \nabla u) = -\nabla p - \rho \nabla \phi$$

from (79), with $p = \rho^2 \frac{\partial e}{\partial \rho}$. For the semidiscrete case, one uses a nonholonomic version of the Euler-Poincaré equations (79), reviewed in the next paragraph.

B Remarks on the nonholonomic Euler-Poincaré variational formulation

Hamilton's principle can be extended to the case in which the system under consideration is subject to a constraint, given by a distribution on the configuration manifold, i.e., a vector subbundle of the tangent bundle. This is known as the Lagrange-d'Alembert principle and, for a system on a Lie group G and constraint $\Delta_G \subset TG$, it is given by the same critical condition (74) but only with respect to variations satisfying the constraint, i.e., $\delta g \in \Delta_G$.

In the G-invariant setting recalled in §A.1 it is assumed that the constraint Δ_G is also G-invariant and thus induces a subspace $\Delta \subset \mathfrak{g}$ of the Lie algebra. In the more general setting of §A.2, one can allow Δ_G to be only G_{a_0}-invariant, although for the situation of interest in this paper, Δ_G is also G-invariant.

The Lagrange-d'Alembert principle yields now the Euler-Poincaré-d'Alembert principle (77) in which we have the additional constraint $u(t) \in \Delta$ on the solution and $v(t) \in \Delta$ on the variations, so that (78) becomes

$$\left\langle\!\!\left\langle \partial_t \frac{\delta \ell}{\delta u}, v \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \frac{\delta \ell}{\delta u}, [u, v] \right\rangle\!\!\right\rangle + \left\langle \frac{\delta \ell}{\delta a}, a \cdot v \right\rangle_V = 0, \quad \text{for all } v \in \Delta, \text{ where } u \in \Delta. \tag{80}$$

In presence of the nonholonomic constraint, (79) becomes

$$\partial_t \frac{\delta \ell}{\delta u} + \mathrm{ad}^*_u \frac{\delta \ell}{\delta u} - \frac{\delta \ell}{\delta a} \diamond a \in \Delta^\circ, \qquad u \in \Delta, \tag{81}$$

where $\Delta^\circ = \{m \in \mathfrak{g}^* \mid \langle\!\langle m, u \rangle\!\rangle = 0, \ \forall\, u \in \Delta\}$.

There are two important remarks concerning (80) and (81) that play an important role for the variational discretization carried out in this paper. First, we note that although the solution belongs to the constraint, i.e., $u \in \Delta$, the equations depend on the expression of the Lagrangian ℓ on a larger space, namely, on $\Delta + [\Delta, \Delta]$. It is not enough to have its expression only on Δ. This is a main characteristic of nonholonomic mechanics. Second, a sufficient condition to get a solvable differential equation is that the map $u \in \Delta \mapsto \frac{\delta \ell}{\delta u} \in \mathfrak{g}^*/\Delta^\circ$ is a diffeomorphism for all a.

C Polynomials

Below we prove two facts about polynomials that are used in the proof of Proposition 3.3. We denote by $H_r(K)$ the space of homogeneous polynomials of degree r on a simplex K. To distinguish powers from indices, we denote coordinates by $x_1, x_2, \ldots, x_n$ rather than $x^1, x^2, \ldots, x^n$ in this section.

Lemma C.1. *Let K be a simplex of dimension $n \geq 1$. For every integer $r \geq 0$,*

$$\left\{ \sum_{i=1}^{N} p_i q_i \mid N \in \mathbb{N},\, p_i, q_i \in P_r(K),\, i = 1, 2, \dots, N \right\} = P_{2r}(K).$$

Proof. This follows from the fact that every monomial in $P_{2r}(K)$ can be written as a product of two monomials in $P_r(K)$. □

Lemma C.2. *Let K be a simplex of dimension $n \in \{2, 3\}$. For every integer $r \geq 0$,*

$$\left\{ \sum_{i=1}^{N} p_i \nabla q_i \mid N \in \mathbb{N},\, p_i, q_i \in P_r(K),\, i = 1, 2, \dots, N \right\} = P_{2r-1}(K)^n.$$

Proof. We proceed by induction.

Denote $Q_r(K) = \left\{ \sum_{i=1}^{N} p_i \nabla q_i \mid N \in \mathbb{N},\, p_i, q_i \in P_r(K),\, i = 1, 2, \dots, N \right\}$. By inductive hypothesis, $Q_r(K)$ contains $Q_{r-1}(K) = P_{2r-3}(K)^n$. It also contains $H_{2r-2}(K)^n$. Indeed, if $fe_k \in H_{2r-2}(K)^n$ with $k \in \{1, 2, \dots, n\}$ and f a monomial, then $f = x_j g$ for some $g \in H_{2r-3}(K)$ and some $j \in \{1, 2, \dots, n\}$, so $fe_k = \sum_i (x_j p_i) \nabla q_i \in Q_r(K)$ for some $p_i, q_i \in P_{r-1}(K)$ by inductive hypothesis. Thus, $Q_r(K)$ contains $P_{2r-2}(K)^n$.

Next we show that $Q_r(K)$ contains every $u \in H_{2r-1}(K)^n$. Without loss of generality, we may assume $u = fe_1$ with $f \in H_{2r-1}(K)$ a monomial. When $n = 2$, the only such vector fields are $u = x_1^a x_2^{2r-1-a} e_1$, $a = 0, 1, \dots, 2r-1$, which can be expressed as

$$x_1^a x_2^{2r-1-a} e_1 = \begin{cases} \frac{1}{r} x_1^{a-r+1} x_2^{2r-1-a} \nabla(x_1^r), & \text{if } a \geq r-1, \\ \frac{1}{a+1} x_2^r \nabla(x_1^{a+1} x_2^{r-1-a}) - \frac{r-1-a}{(a+1)r} x_1^{a+1} x_2^{r-1-a} \nabla(x_2^r), & \text{if } a < r-1. \end{cases}$$

The case $n = 3$ is handled similarly by considering the vector fields fe_1 with

$$f \in \{x_1^a x_2^b x_3^{2r-1-a-b} \mid a, b \geq 0,\, a + b \leq 2r-1\}.$$

□

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