The Novikov Conjecture for Linear Groups

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Abstract

Let K be a field. We show that every countable subgroup of $GL(n, K)$ is uniformly embeddable in a Hilbert space. This implies that Novikov's higher signature conjecture holds for these groups. We also show that every countable subgroup of $GL(2, K)$ admits a proper, affine isometric action on a Hilbert space. This implies that the Baum-Connes conjecture holds for these groups. Finally, we show that every subgroup of $GL(n, K)$ is exact, in the sense of C^*-algebra theory.

Introduction

The purpose of this paper is to prove Novikov's higher signature conjecture for linear groups, that is, for subgroups of the general linear group of a field:

Theorem. *Let M be a smooth, closed and oriented manifold, let n be any positive integer, and let $\rho\colon \pi_1(M) \to GL(n, K)$ be a homomorphism from the fundamental group of M into the general linear group of a field. If $c \in H^*(GL(n, K), \mathbb{Q})$ is any cohomology class then the higher siganture*

$$\mathrm{Sign}_c(M) = \langle L(M) \cup \rho^*(c), [M] \rangle$$

is an oriented homotopy invariant of M.

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The group $GL(n, K)$ in the statement should be viewed as a *discrete* group.

The Novikov conjecture in the case where the image of ρ is a closed subgroup of $GL(n, \mathbb{R})$ was settled by Kasparov [20]. Subsequently Kasparov and Skandalis dealt with (products of) linear groups over local fields [21]. The main difficulty in extending these results to the present setting is visible in very simple examples. The subgroup of $GL(2, \mathbb{R})$ generated by $\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ contains an infinite-rank, free abelian subgroup and therefore admits no properly discontinuous action on a finite dimensional complex. As a result, the methods developed by Kasparov and Skandalis do not readily apply. However this particular subgroup is solvable, and in particular amenable, and it was shown by Higson and Kasparov [19] how to deal with arbitrary (countable) amenable groups by replacing finite-dimensional complexes with infinite-dimensional Hilbert spaces.[1] To prove the theorem, we shall show that *every* countable subgroup of $GL(n, K)$ admits a uniform embedding into a Hilbert space, in the sense of Gromov [13]. According to recent work in C^*-algebra K-theory (see [17], [28], [32]), if a discrete group Γ is uniformly embeddable into Hilbert space, and if A is any Γ-C^*-algebra, then the Baum-Connes assembly map

$$\mu_A \colon K_*^{\Gamma}(\mathcal{E}\Gamma; A) \to K_*(C^*_{\text{red}}(\Gamma, A)) \tag{0.1}$$

is split injective. Finally, injectivity of the Baum-Connes assembly map implies the Novikov conjecture (see [5]).

The topic of uniform embedding into Hilbert space is closely related to the C^*-algebraic notion of *exactness* of group C^*-algebras [29]. For example, every countable exact group is uniformly embeddable (the converse is not known, although there is at present no known example of a uniformly embeddable group which is not exact). We shall exploit this relationship to prove the following theorem:

Theorem. *Let K be a field and let n be a positive integer. The reduced group C^*-algebra of every subgroup of $GL(n, K)$ is exact.*

Our construction of uniform embeddings of subgroups of $GL(n, K)$ uses some elementary properties of valuations on fields. In the special case of subgroups of $GL(2, K)$ we use more specialized properties of real hyperbolic space and of the tree associated to a discrete valuation to prove the following stronger result.

Theorem. *Let K be a field. The Baum-Connes assembly map* (0.1) *is an isomorphism for any countable subgroup of $GL(2, K)$ and any A.*

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[1] An alternate argument is based on the work of Pimsner and Voiculescu [26].

Indeed we show that a group Γ as in the statement admits a metrically proper, isometric action on a Hilbert space (Higson and Kasparov showed in [18, 19] that if Γ is a group which admits such an action then the Baum-Connes assembly map (0.1) is an isomorphism for any A). Zimmer showed that every property T subgroup of $GL(2, \mathbb{C})$ is necessarily finite [33]. Our theorem improves on this, since every property T group which admits a proper, isometric action on Hilbert space is necessarily finite

In the final section we apply our results on the Novikov conjecture to the problem of homotopy invariance of relative eta invariants. The following theorem strengthens an earlier result of the third author [31].

Theorem. *Let M and M' be homotopy equivalent smooth, closed, oriented, odd-dimensional manifolds with fundamental group π and let $\rho : \pi \to U(k)$ be a finite dimensional unitary representation. The difference $\widetilde{\eta}_\rho(M) - \widetilde{\eta}_\rho(M')$ lies in the subring of $\mathbb{Q}$ generated by $\mathbb{Z}$, the inverses of the orders of torsion elements in $\rho[\pi]$ and $1/2$.*

For an interesting and related analysis of linear groups via valuation theory the reader is referred to Alperin and Shalen [1].

1 Valuations

In this section we record some elementary facts concerning fields and valuations; a basic reference for this material is [8]. Let K be a field. For our purposes, a *valuation* on K is a map $d \colon K \to [0, \infty)$ such that

(i) $d(x) = 0 \quad \Leftrightarrow \quad x = 0$,

(ii) $d(xy) = d(x)d(y)$,

(iii) $d(x + y) \leq d(x) + d(y)$.

A valuation is *archimedean* if there is an embedding of K into $\mathbb{C}$ for which the valuation is given by the formula

$$d(x) = |x|, \quad \text{for all } x \in K$$

(this is in slight variance with the reference [8], which also allows the formula $d(x) = |x|^\alpha$). A valuation is *discrete* if the triangle inequality (iii) can be replaced by the stronger inequality (iii)′ $d(x+y) \leq \max\{d(x), d(y)\}$,

and if, in addition, the range of d on $K^\times$ is a discrete subgroup of the multiplicative group of positive real numbers. If d is a discrete valuation on K, then the set

$$\mathcal{O} = \{x \in k : d(x) \leq 1\} \subseteq K$$

is a subring of K, called the *ring of integers* in K, and the set

$$\mathfrak{m} = \{x \in K : d(x) < 1\}$$

is a principal ideal of $\mathcal{O}$. A generator π of $\mathfrak{m}$ is called a *uniformiser*. If we set aside the *trivial valuation*, for which $d \equiv 1$ on $K^\times$, then the function

$$\nu(x) = \frac{\log(d(x))}{\log(d(\pi))}, \quad \text{for } x \in K^\times$$

is a discrete valuation in the sense of commutative algebra; in other words ν is a surjective, integer-valued function on $K^\times$ satisfying

(iv) $\nu(xy) = \nu(x) + \nu(y)$,

(v) $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$ (one sets $\nu(0) = \infty$).

If R is a unique factorization domain, and if $p \in R$ is prime, then the formula

$$d(p^n \frac{a}{b}) = 2^{-n}, \quad (p, a) = (p, b) = 1$$

defines a nontrivial discrete valuation on the field of fractions of R. (The number 2 could be replaced by any number greater than 1.)

If R is an integral domain, and if d is a non-negative, real-valued function on R satisfying the axioms (i), (ii) and (iii)′ of a discrete valuation, then d extends uniquely to a valuation on the fraction field of R; the extension is given by

$$d\left(\frac{r}{s}\right) = \frac{d(r)}{d(s)}.$$

Henceforth we will use this fact without mention.

If K is a subfield of a finitely generated field L, and if d is a valuation on K, then d extends to a valuation on L (usually in more than one way).

2 Discrete Embeddability

The purpose of this section is to show that every (finitely generated) field has plenty of valuations, in the following sense:

2.1 Definition. A finitely generated field K is *discretely embeddable* if, for every finitely generated subring $R \subseteq K$, there is a countable family $\{d_1, d_2, \dots\}$ of valuations on K, each either archimedean or discrete, with the property that if $N_1, N_2, \dots$ are any positive numbers then the set

$$\{ r \in R : \ d_j(r) < N_j, \text{ for all } j \}$$

is finite. Given a subring $R \subseteq K$, a family of valuations with this property is called R-*proper*.

2.2 Theorem. *Every finitely generated field is discretely embeddable.*

Proof. This follows from the subsequent Lemmas 2.3, 2.4 and 2.5 by an obvious induction argument. □

2.3 Lemma. *Finite fields and the rational number field are discretely embeddable.*

Proof. The result is trivial for finite fields. As for $\mathbb{Q}$, the countable family of valuations consisting of the unique archimedean valuation, together with the p-adic valuations

$$d(p^n \frac{a}{b}) = p^{-n}, \quad (p, a) = (p, b) = 1,$$

is R-proper, for every finitely generated subring $R \subseteq \mathbb{Q}$. □

2.4 Lemma. *If a field K is discretely embeddable then so is any extension $K(X)$ which is generated by a single transcendental element.*

Proof. Let $S \subseteq K(X)$ be a finitely generated subring. There is a finitely generated subring $R \subseteq K$, and there are finitely many monic irreducible polynomials $p_i \in K[X]$, with coefficients in R, such that S is included in the ring obtained from $R[X]$ by inverting the elements p_i.

Let $\{d_j\}$ be an R-proper family of valuations on K. We extend each valuation d_j to $K(X)$, as follows. If d_j is discrete then we employ the formula

$$d_j(a) = \max_k \{ d_j(a_k) \}, \quad a = a_0 + a_1 X + \cdots + a_n X^n \in K[X].$$

If d_j is archimedean, corresponding to an embedding $K \subseteq \mathbb{C}$, then we extend this to an embedding of $K(X)$ into $\mathbb{C}$ in countably many distinct ways, and extend d_j accordingly to countably many valuations d_{ij} on $K(X)$.

To the collection of all these extended valuations we add the valuation defined by the formula

$$d_\infty(a) = 2^{\deg(a)}, \quad a \in K[X],$$

along with the valuations

$$d_{p_i}(p_i^n \frac{a}{b}) = 2^{-n}$$

associated to the primes $p_i \in K[X]$. We claim that the countable family of valuations that we have now assembled is S-proper. Suppose that $s \in S$ satisfies bounds $d(s) < N_d$, as in Definition 2.1. Every element of S has the form

$$s = \frac{a}{p_{i_1}^{k_{i_1}} \cdots p_{i_l}^{k_{i_l}}},$$

where $a \in R[X]$, no p_{i_j} divides a and the $k_{i_j} > 0$. We see right away by considering the valuations d_{p_i} that the degrees $k_{i_1}, \ldots, k_{i_l}$ are bounded. Using d_∞ we then see that the degree of numerator a is bounded, by say m. If d_j was one of the discrete valuations on K with which we started, then $d_j(a)$ is bounded, which means that if a_k is a coefficient of the polynomial a, then $d_j(a_k)$ is bounded. Suppose d_j is one of the archimedean valuations on K with which we started, determined by an inclusion $K \subseteq \mathbb{C}$, and suppose that the extensions d_{ij} of d_j to $K(X)$ are determined by extensions of this inclusion which send X to t_i. The values $d_{ij}(a)$ are bounded, which means that the $m+1$ complex numbers

$$b_i = a_0 + a_1 t_i + a_2 t_i^2 + \cdots + a_m t_i^m \qquad (i = 1, \ldots, m+1)$$

are bounded. Solving for the a_i in terms of the b_i we see that the a_i are bounded too.

We have shown that each d_j in the original R-proper family, applied to each coefficient of the polynomial a, is bounded. As a result, it follows from the definition of an R-proper family that the coefficients of the polynomial a belong to a finite subset of R. Therefore the set of possible numerators a is finite, and we conclude that the set of all $s \in S$ satisfying bounds $d(s) < N_d$ is finite, as required. □

2.5 Lemma. *Every finite extension of a discretely embeddable field is discretely embeddable.*

Proof. Let K be discretely embeddable and let L be a finite extension of K. We must show that L is discretely embeddable. Since a subfield of a discretely embeddable field is itself discretely embeddable, by enlarging L if necessary, we may assume that L is a finite normal extension of K.

Let S be a finitely generated subring of L. Fix a basis $\{x_1, \ldots, x_n\}$ for L over K. Multiplication by $s \in S$ is an endomorphism of the K-vector space L which is represented with respect to the basis $\{x_i\}$ as a matrix with elements from K. Let $R \subseteq K$ be the subring generated by the (finitely many) matrix elements of a finite generating set for S. Let $\{d_j\}$ be an R-proper family of valuations on K. Each valuation d_j on K admits between 1 and n extensions d_{ij} to L. We show that for all positive numbers N_j the set

$$\mathcal{S} = \{\, s \in S : \; d_{ij}(s) < N_j \,\}$$

is finite. Observe that the collection of valuations $\{d_{ij}\}$ is stable under the action of the Galois group $\mathrm{Gal}(L|K)$ (this is because if d_{ij} is a valuation on L which extends a given valuation d_j on K, then so is the composition of d_{ij} with any automorphism of L over K). Consequently, if $s \in \mathcal{S}$, and if s' is a conjugate of s under $\mathrm{Gal}(L|K)$, then s' satisfies the inequalities defining $\mathcal{S}$ too.

Let $s \in \mathcal{S}$. Because of the way R is defined, the coefficients of the characteristic polynomial of s (considering s as an endomorphism of the K-vector space L by multiplication) are elements of R. The roots of this polynomial are s and its conjugates (each counted with some multiplicity between 1 and n). Since the coefficients are elementary symmetric functions of the roots, it follows that the coefficients $r \in R$ satisfy inequalities of the form

$$d_j(r) < M_j,$$

where M_j is some function of the N_j and n. It follows that if $s \in \mathcal{S}$ then the coefficients of the characteristic polynomial of s lie in a finite set. As a result, the number of different characteristic polynomials is finite. Hence the number of distinct roots of these polynomials is finite, and so the set $\mathcal{S}$ is finite, as required. □

3 Hilbert Space Preliminaries

Our proofs of the Novikov and Baum-Connes conjectures will rely on the notions of uniform embeddability and a-T-menability, respectively. In this section we shall recall the basic definitions.

3.1 Definition. A discrete group Γ is *uniformly embeddable* (into Hilbert space) if there is a function $f\colon \Gamma \to \mathcal{H}$ such that:

(i) For every finite set $F \subseteq \Gamma$ there is a constant $A_F > 0$ such that

$$g_1^{-1}g_2 \in F \quad \Rightarrow \quad \|f(g_1) - f(g_2)\| < A_F.$$

(ii) For every $A > 0$ there exists a finite set $F_A \subseteq \Gamma$ such that

$$\|f(g_1) - f(g_2)\| < A \quad \Rightarrow \quad g_1^{-1}g_2 \in F_A.$$

The function f is a *uniform embedding* (even though it need not be one-to-one).

Remark. A uniformly embeddable discrete group is necessarily countable.

Remark. In the case of a finitely generated group Γ it suffices to check condition (i) on a finite generating set $S \subseteq \Gamma$. Indeed if $F \subseteq \Gamma$ is finite, then there exists k such that every element of F can be written as a product of at most k elements from S, and it follows easily from the triangle inequality that we may take $A_F = kA_S$ in condition (i).

3.2 Definition. Let G be a group. A *length function* on G is function $\ell : G \to [0, \infty)$ such that

(i) $\ell(e) = 0$,

(ii) $\ell(g) = \ell(g^{-1})$, and

(iii) $\ell(g_1g_2) \leq \ell(g_1) + \ell(g_2)$.

We do *not* require that ℓ be proper, nor do we require that if $\ell(g) = 1$ then $g = e$.

3.3 Definition. A group G with length function ℓ is ℓ-*uniformly embeddable* (into Hilbert space) if there is a function $f : G \to \mathcal{H}$ such that

(i) For every $B > 0$ there is a constant $A_B > 0$ such that

$$\ell(g_1^{-1}g_2) < B \quad \Rightarrow \quad \|f(g_1) - f(g_2)\| < A_B.$$

(ii) For every $A > 0$ there exists $B_A > 0$ such that

$$\|f(g_1) - f(g_2)\| < A \quad \Rightarrow \quad \ell(g_1^{-1}g_2) < B_A.$$

The function f is an *ℓ-uniform embedding.*

Remark. Uniform embeddability is equivalent to ℓ-uniform embeddability for a single *proper* length function ℓ. Also, ℓ-uniform embeddability (as defined above) is equivalent to uniform embeddability (as defined by Gromov [13]) with respect to the left invariant pseudo-metric defined by ℓ.

There are various equivalent formulations of the condition of uniform embeddability, and it is convenient to work with some of them in this paper. We shall rely primarily on Propositions 3.7 and 3.8. For a similar discussion see [14]; for a different perspective on the proofs of these propositions see [10].

3.4 Definition. Let X be a set. A function $\delta\colon X \times X \to \mathbb{R}$ is a *negative-type kernel* on X if

(i) $\delta(x, x) = 0$, for every $x \in X$,

(ii) $\delta(x_1, x_2) = \delta(x_2, x_1)$, for every $x_1, x_2 \in X$, and

(iii) if $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \Rightarrow \quad \sum_{i,j=1}^{n} \lambda_i \lambda_j \delta(x_i, x_j) \leq 0.$$

If $f\colon X \to \mathcal{H}$ is *any* function from a set X into a Hilbert space then the function

$$\delta(x_1, x_2) = \|f(x_1) - f(x_2)\|^2 \tag{3.1}$$

is a negative-type kernel. Conversely if $\delta\colon X \times X \to \mathbb{R}$ is a negative-type kernel then there is an essentially unique Hilbert space function $f\colon X \to H$ which is related to δ as in equation (3.1). See [11]. As a result of this, it is easy to reformulate the definition of uniform embeddability in terms of negative-type kernels:

3.5 Proposition. *A group Γ is ℓ-uniformly embeddable into Hilbert space if and only if there exists a negative type kernel $\delta\colon \Gamma \times \Gamma \to \mathbb{R}$ with the following properties:*

(i) *For every $B > 0$ there is some $A_B > 0$ such that*

$$\ell(g_1^{-1} g_2) \leq B \quad \Rightarrow \quad \delta(g_1, g_2) \leq A_B.$$

(ii) *For every* $A > 0$ *there is some* $B_A > 0$ *such that*

$$\delta(g_1, g_2) \leq A \quad \Rightarrow \quad \ell(g_1^{-1}g_2) \leq B_A.$$

□

Uniform embeddability can be further characterized in terms of kernels which are *positive-definite*, in the sense of the following definition.

3.6 Definition. A function $\phi\colon X \times X \to \mathbb{R}$ is a (real-valued, normalized) *positive-definite kernel* on X if

(i) $\phi(x, x) = 1$, for every $x \in X$,

(ii) $\phi(x_1, x_2) = \phi(x_2, x_1)$, for every $x_1, x_2 \in X$, and

(iii) if $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and $x_1, \ldots, x_n \in X$, then

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \phi(x_i, x_j) \geq 0.$$

Remark. A positive-definite kernel ϕ automatically satisifies $\phi(x, y) \leq 1$.

A kernel $\delta(x_1, x_2)$ is of negative-type if and only if the kernels

$$\phi_t(x_1, x_2) = e^{-t\delta(x_1, x_2)}$$

are positive-definite, for all $t > 0$ (this is Schoenberg's Theorem; see for example [11]). In the other direction, if ϕ is a positive-definite kernel then $\delta(x_1, x_2) = 1 - \phi(x_1, x_2)$ is a negative-type kernel. Using these facts we obtain the following:

3.7 Proposition. *A group* G *with length function* ℓ *is* ℓ*-uniformly embeddable into Hilbert space if and only if, for every* $\varepsilon > 0$ *and every* $A > 0$*, there is a positive-definite kernel* $\phi\colon G \times G \to \mathbb{R}$ *with the following properties:*

(i) $\ell(g_1^{-1}g_2) < A \quad \Rightarrow \quad |1 - \phi(g_1, g_2)| < \varepsilon.$

(ii) *For every* $\delta > 0$ *there is a* $B > 0$ *such that*

$$|\phi(g_1, g_2)| \geq \delta \quad \Rightarrow \quad \ell(g_1^{-1}g_2) < B. \qquad \square$$

Specializing to the case of a proper length function we obtain the following:

3.8 Proposition. *A countable discrete group* Γ *is uniformly embeddable into Hilbert space if and only if for every* $\varepsilon > 0$ *and every finite set* F *there is a positive-definite kernel* $\phi\colon \Gamma \times \Gamma \to \mathbb{R}$ *with the following properties:*

(i) $g_1^{-1}g_2 \in F \quad \Rightarrow \quad |1 - \phi(g_1, g_2)| < \varepsilon.$

(ii) *For every* $\delta > 0$ *there is a finite set* $F_\delta \subseteq \Gamma$ *such that*

$$|\phi(g_1, g_2)| \geq \delta \quad \Rightarrow \quad g_1^{-1}g_2 \in F_\delta. \qquad \square$$

It is useful to consider kernels on a group G which which are G-invariant, meaning that $k(gg_1, gg_2) = k(g_1, g_2)$, for all $g, g_1, g_2 \in G$. A G-invariant kernel $k(g_1, g_2)$ determines a one-variable fundtion $k(g) = k(1, g)$, which in turn determines the kernel by the formula $k(g_1, g_2) = k(g_1^{-1}g_2)$. A function on G is *negative-type* or *positive-definite* if the associated G-invariant kernel is. These functions are related to group actions on Hilbert space, as follows:

3.9 Lemma. *Let* G *be a group. A function* $\delta\colon G \to \mathbb{R}$ *is of negative-type if and only if there exists an affine-isometric action of* G *on* $\mathcal{H}$*, and a vector* $v \in H$*, such that*

$$\delta(g) = \|g \cdot v - v\|^2, \quad \forall g \in G.$$

A function $\phi\colon G \to \mathbb{R}$ *is positive-definite if and only if there exists an isometric linear action of* G *on* $\mathcal{H}$ *and a unit vector* $v \in H$ *such that*

$$\phi(g) = \langle g \cdot v, v\rangle, \quad \forall g \in G. \qquad \square$$

3.10 Definition. A discrete group Γ is *a-T-menable* or has the *Haagerup property* if there exists an affine-isometric action of Γ on $\mathcal{H}$ with the property that if $v \in \mathcal{H}$ then $\lim_{g\to\infty} \|g \cdot v\| = \infty$. An action with this property is *metrically proper*.

The following characterization of a-T-menability follows immediately from the first part of Lemma 3.9. For details and additional information consult [9].

3.11 Proposition. *A discrete group* Γ *is a-T-menable if and only if there exists a proper, negative-type function* $\delta\colon \Gamma \to \mathbb{R}$. $\square$

4 Uniform Embeddability of Linear Groups

In this section we shall prove the following theorem:

4.1 Theorem. *Let* K *be a field. Every countable subgroup of* $GL(n, K)$ *is uniformly embeddable into Hilbert space.*

Appealing to results of Skandalis, Tu and Yu [17, 28] we conclude, as described in the Introduction, that the higher signatures associated to a homomorphism $\rho\colon \pi_1(M) \to G$ are homotopy invariants whenever the image of ρ is uniformly embeddable into Hilbert space. We thereby obtain from Theorem 4.1 the first theorem of the Introduction. Moreover, we have the following theorem:

4.2 Theorem. *Let* K *be a field and let* Γ *be a countable subgroup of* $GL(n, K)$. *The Baum-Connes assemby map* (0.1) *is split injective for every coeffecient* $\Gamma - C^*$-*algebra* A.

A countable discrete group Γ is uniformly embeddable if and only if all its finitely generated subgroups are (this follows from Proposition 3.8; compare [10]). In proving Theorem 4.1 we may therefore assume that Γ is finitely generated. Having done so we may assume that the field K is finitely generated.

Thus, *we shall now assume that* K *is a finitely generated field and that* Γ *is a finitely generated subgroup of* $GL(n, K)$. To construct the required uniform embedding of Γ we shall first construct many embeddings of $GL(n, K)$ into Hilbert space which are uniform with respect to valuations on K, in a sense which we now make precise.

Let d be a discrete valuation on K. If $g = [g_{ab}]$ is a matrix in $GL(n, K)$ and if $[g^{ab}]$ denotes the inverse matrix then the formula

$$\ell_d(g) = \log \max_{a,b}\{d(g_{ab}), d(g^{ab})\} \tag{4.1}$$

defines a length function on $GL(n, K)$. If d is an archimedean valuation on K (coming from an embedding of K into $\mathbb{C}$) then the formula

$$\ell_d(g) = \log \max\{\|g\|, \|g^{-1}\|\}, \tag{4.2}$$

which involves the usual operator norm of a matrix in $GL(n, \mathbb{C})$, defines a length function.

4.3 Definition. Let d be a discrete or archimedean valuation on K and let ℓ_d be the associated length function on $GL(n, K)$, given by (4.1) or (4.2). A d-*uniform embedding* of a group $G \subseteq GL(n, K)$ is an ℓ_d-uniform embedding of G in the sense of Definition 3.3.

4.4 Proposition. *If* d *is an archimedean valuation on* K *then there exists a* d*-uniform embedding of* $GL(n, K)$ *into Hilbert space.*

Proof. The length function we are using is the restriction to $GL(n, K)$ of a length function on $GL(n, \mathbb{C})$ via an embedding $K \subseteq \mathbb{C}$. Therefore, it suffices to show that $GL(n, \mathbb{C})$ is d-uniformly embeddable.

The group $G = GL(n, \mathbb{C})$ may be written as a product $G = PH$, where $H = U(n)$ (a maximal compact subgroup of G) and P is the group of upper triangular matrices with positive diagonal entries. The length function ℓ_d is bi-H-invariant in the sense that

$$\ell_d(h_1 g h_2) = \ell_d(g), \quad \text{for all } h_1, h_2 \in H \text{ and } g \in G. \tag{4.3}$$

As a consequence, the function $g = ph \mapsto p$ mapping $G \to P$ is isometric in the sense that if $g_1 = p_1 h_1$ and $g_2 = p_2 h_2$ then $\ell_d(g_1^{-1} g_2) = \ell_d(p_1^{-1} p_2)$. It follows that the formula $f(ph) = f(p)$ extends a d-uniform uniform embedding f of P to one of G. Indeed, G is d-uniformly embeddable if and only if P is.

Finally, it is well known how to d-uniformly embed the solvable group P (compare [6]). Since P is amenable there is a sequence $\{\phi_m\}$ of compactly supported, positive-definite functions on P which converges to 1 uniformly on compact sets. Now, the length function ℓ_d on G, and also on P, has the property that bounded subsets are precisely those with compact closure. Combined, these observations show, according to Proposition 3.7, that P is d-uniformly embeddable. □

The case of discrete valuations is just a little more complicated. Before dealing with it we make some preliminary observations.

Let K be a field, let d be a discrete valuation on K and let π be a uniformiser. Let $G = GL(n, K)$. We define several subgroups of G. Let H be the subgroup consisting of those matrices g for which the entries of both g and g^{-1} belong to the ring of integers $\mathcal{O}$; let A be the subgroup of diagonal matrices whose diagonal entries are integer powers of the uniformiser π; let N be the subgroup comprised of the unipotent upper triangular matrices (that is, their diagonal entries are all 1); let $P = AN$, which is again a subgroup of G.

4.5 Lemma. $G = PH$.

Proof. The decomposition is accomplished using elementary column operations, taking care that only $\mathcal{O}$-multiples of one column are added to other columns. Let $g \in G$. Apply an exchange of columns operation to put into the (n, n) position an element x whose valuation is maximal along the nth row. Every element of this row is then an $\mathcal{O}$-multiple of x, so we can then add appropriate integer multiples of the last column to the other columns to clear the other entries of the last row. Having done so, we obtain a decomposition

$$g = \begin{pmatrix} \star & \star & \dots & \star & \star \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & \star & \dots & \star & \star \\ 0 & 0 & \dots & 0 & x \end{pmatrix} \cdot h,$$

where $h \in H$. Now repeat the process on the upper $(n-1) \times (n-1)$ block of gh^{-1}, and continue. After having eventually obtained h for which gh^{-1} is upper triangular, a final right-multiplication by a diagonal matrix in H will reduce each diagonal entry of the upper triangular matrix to a power of π. □

4.6 Lemma. *There is a sequence* $\{\phi_m\}_{m=1}^{\infty}$ *of positive-definite functions on* P *such that:*

(i) *For all* $C > 0$ *and all* $\varepsilon > 0$ *there exists* M_C *such that for all* $g \in P$

$$m > M_C \quad \textit{and} \quad \ell_d(g) < C \quad \Rightarrow \quad |\phi_m(g) - 1| < \varepsilon.$$

(ii) *For all* m *there exists* M *such that for all* $g \in P$

$$\ell_d(g) > M \quad \Rightarrow \quad \phi_m(g) = 0.$$

Proof. We shall construct the required positive definite functions as matrix coeffecients of quasiregular representations of P.

Let N_m be the subgroup of N generated by the elements of length not greater than m. We claim that N_m is bounded. Indeed, it is contained in the subgroup of N consisting of those matrices $[g_{ab}]$ for which

$$g_{ab} \in \pi^{(a-b)m'}\mathcal{O},$$

provided $m' \geq m \cdot |\log d(\pi)|^{-1}$. Moreover, every element of this subgroup has length $\leq nm' \cdot |\log d(\pi)|$.

Let P act on $\ell^2(P/N_m)$ by the quasiregular representation. Denote by A_m the finite subset of the diagonal group A consisting of those matrices a for which

$\ell_d(a) \leq \frac{m}{4}$. Let $\nu_m \in \ell^2(P/N_m)$ be the normalized characteristic function of A_m. Finally, define

$$\phi_m(g) = \langle \nu_m, g \cdot \nu_m \rangle_{\ell^2(P/N_m)}.$$

We check that the sequence $\{\phi_m\}$ has the required properties.

For the first claim in the proposition it suffices to show that

(iii) If $n \in N$ and $\ell_d(n) \leq C$ then $n \cdot \nu_m = \nu_m$ for all $m > 2C$.

(iv) $\forall\, C > 0\ \ \phi_m(a) \to 1$ uniformly on $\{\, a \in A \colon \ell_d(a) < C \,\}$.

Indeed, assuming these let $g = an \in P$ with $\ell_d(g) \leq C$. We then have

$$\ell_d(a) \leq \ell_d(g) \quad \text{and} \quad \ell_d(n) \leq \ell_d(a) + \ell_d(g) \leq 2C.$$

It follows from (iii) that $\phi_m(g) = \phi_m(a)$ for $m > 4C$. The first claim in the proposition now follows easily from (iv).

The proofs of (iii) and (iv) are straightforward. For (iii) we show that such n fixes every coset appearing in ν_m. Indeed, if $a \in A_m$ then $naN_m = a(a^{-1}na)N_m$ so that the coset aN_m is fixed if $a^{-1}na \in N_m$. But,

$$\ell_d(a^{-1}na) \leq 2\ell_d(a) + \ell_d(n) \leq \tfrac{m}{2} + C \leq m.$$

Item (iv) amounts to the fact that an increasing sequence of balls gives a Følner sequence for the amenable group $A \cong \mathbb{Z}^n$; note that $a \in A_m$ if and only if the diagonal entries of a are of the form π^k with $|k \cdot \log d(\pi)| \leq \frac{m}{4}$.

For the final claim in the proposition fix m. We show that if $g = an \in P$ is such that $\phi_m(g) \neq 0$ then $\ell_d(g) \leq m + \operatorname{diam}(N_m)$. Indeed, if $\phi_m(g) \neq 0$ there exists $a_1 \in A_m$ such that ga_1N_m represents a coset appearing in ν_m. We have

$$ga_1N_m = aa_1(a_1^{-1}na_1)N_m, \quad \text{with } aa_1 \in A \text{ and } a_1^{-1}na_1 \in N.$$

If follows that $a \in A_m a_1^{-1}$ and $n \in a_1 N_m a_1^{-1}$. Hence

$$\ell_d(g) \leq \ell_d(a) + \ell_d(n) \leq \frac{m}{2} + \frac{m}{2} + \operatorname{diam}(N_m). \qquad \square$$

4.7 Proposition. *If d is a discrete valutation on K then there exists a d-uniform embedding of $G = GL(n, K)$ into Hilbert space.*

Proof. Let K be a field with a discrete valuation. The length function we are using is bi-H-invariant in the sense of (4.3). Indeed, if $h \in H$ then $\ell_d(h) = 0$ since the entries of h and h^{-1} are all in $\mathcal{O}$ and hence each has $d \leq 1$. It follows that for h_1, $h_2 \in H$ and $g \in G$ we have

$$\ell_d(h_1 g h_2) \leq \ell_d(h_1) + \ell_d(g) + \ell_d(h_2) = \ell_d(g).$$

The reverse inequality follows similarly.

As a consequence the map $G \to P$, obtained by fixing, for each $g \in G$ a decomposition $g = ph$ and assigning $g = ph \mapsto p$, is isometric and G is d-uniformly embeddable if and only if P is.

Finally, P is d-uniformly embeddable. Indeed, the sequence of positive-definite functions $\{\phi_m\}$ constructed in Lemma 4.6 lift to positive-definite kernels on P that satisfy the conditions of Proposition 3.7. □

Proof of Theorem 4.1. Let K be a finitely generated field and let Γ be a finitely generated subgroup of $GL(n, K)$. (We reduced to this case earlier.) Fix a finite, symmetric generating set for Γ. According to Theorem 2.2 the field K is discretely embeddable. Let $R \subseteq K$ be the ring generated by the matrix entries of the elements of Γ. Observe that R is a finitely generated ring and let $\{d_j\}$ be an R-proper family of valuations on K. Let f_j be a d_j-uniform embedding of $GL(n, K)$ into a Hilbert space $\mathcal{H}_j$. We shall build a uniform embedding of Γ as an appropriate weighted sum of the f_j.

According to Definition 3.3 there exist $A_j > 0$ such that $\|f_j(g_1) - f_j(g_2)\| < A_j$ whenever $g_1^{-1} g_2$ is a generator. Choose a sequence $\{\varepsilon_j\}$ of positive numbers with the property that

$$\sum_j \varepsilon_j^2 \|f_j(s)\|^2 < \infty, \tag{4.4}$$

for every generator s, and such that $\sum_j \varepsilon_j^2 A_j^2 < \infty$. Suppose now that an element $g \in G$ is a k-fold product of generators, say $g = s_1 \cdots s_k$. Then

$$\begin{aligned} \|f_j(g) - f_j(s_1)\| &= \|f_j(s_1 \cdots s_k) - f(s_1)\| \\ &\leq \|f_j(s_1 \cdots s_k) - f_j(s_1 \cdots s_{k-1})\| + \cdots + \|f_j(s_1 s_2) - f_j(s_1)\| \\ &\leq (k-1) A_j. \end{aligned}$$

It follows easily that the inequality (4.4) holds not just for every generator s but for every $g \in \Gamma$, and we can define a map f from Γ into the direct sum Hilbert space $\oplus \mathcal{H}_j$ by the formula

$$f(g) = \oplus \varepsilon_j f_j(g).$$

The function f is the required uniform embedding.

Let us check that f satisfies the conditions of Definition 3.1. To verify item (i) it suffices to consider the case when $g_1^{-1}g_2$ is a generator. In this case we have

$$\|f(g_1) - f(g_2)\|^2 = \sum_j \varepsilon_j^2 \|f_j(g_1) - f_j(g_2)\|^2 \leq B = \sum_j \varepsilon_j^2 A_j^2.$$

To verify item (ii) let $A > 0$ and suppose that $\|f(g_1) - f(g_2)\| < A$. We then of course have $\|f_j(g_1) - f_j(g_2)\| < \varepsilon_j^{-1}A$, for every j. Since f_j is a d_j-uniform embedding, it follows that there exist constants B_j such that $\ell_j(g_1^{-1}g_2) < B_j$, for every j. This means, in particular, that the entries of the matrix $g_1^{-1}g_2$ are d_j-bounded, for every j, and hence belong to a finite set in the ring R. Hence $g_1^{-1}g_2$ belongs to a finite subset of Γ, as required. □

5 The Haagerup Approximation Property

We are going to strengthen the main theorems of the last section, as they apply to $GL(2, K)$:

5.1 Theorem. *Let* K *be a field. Every countable subgroup of* $GL(2, K)$ *has the Haagerup property.* □

Higson and Kasparov showed that the Baum-Connes conjecture holds for groups with the Haagerup property [18, 19]. Therefore we obtain the following theorem:

5.2 Theorem. *Let* K *be a field. Every countable subgroup of* $GL(2, K)$ *satisfies the Baum-Connes conjecture.*

A countable discrete group has the Haagerup property if and only if all of its finitely generated subgroups do [9]. In proving Theorem 5.1 we may therefore assume that Γ is finitely generated. Having confined our attention to finitely generated Γ we may assume that the field K is finitely generated.

Moreover it suffices to consider the case of subgroups of $SL(2, K)$. Indeed, if $\Gamma \subseteq GL(2, K)$ then $\Gamma \cap SL(2, K)$ is a normal subgroup of Γ with abelian quotient. Since the class of groups with the Haagerup property is closed under extensions with amenable quotient [9] Γ has the Haagerup property if $\Gamma \cap SL(2, K)$ does.

In light of these remarks *we assume that* K *is a finitely generated field and that* Γ *is a finitely generated subgroup of* $SL(2, K)$. Our strategy for proving the

Haagerup property is to build a proper negative-type function on Γ from an appropriate family of negative-type functions, each one obtained from a valuation on K. The individual functions comprising the family will be geometric in origin.

The following lemma is essentially due to Haagerup [15]; for a detailed proof see [11]. For a proof of the second lemma see [11].

5.3 Lemma. *Let T be a simplicial tree and let*

$$\mathrm{distance}_T(v_1, v_2) = \begin{cases} \textit{the number of edges on the shortest} \\ \textit{edge path in } T \textit{ from } v_1 \textit{ to } v_2. \end{cases}$$

Let G be a group acting by isometries on T. For every vertex v in T the function

$$\delta(g) = \mathrm{distance}_T(v, g \cdot v)$$

is of negative-type on G. □

5.4 Lemma. *Let X be the symmetric space $SL(2,\mathbb{C})/SU(2)$ (namely 3-dimensional real hyperbolic space), equipped with the unique (up to overall scale factor) $SL(2,\mathbb{C})$-invariant Riemannian structure. Let*

$$\mathrm{distance}_X(x_1, x_2) = \begin{cases} \textit{length of the shortest path} \\ \textit{in } X \textit{ from } x_1 \textit{ to } x_2. \end{cases}$$

Let G be a group acting by isometries on X. For every point $x \in X$ the function

$$\delta(g) = \mathrm{distance}_X(x, g \cdot x)$$

is of negative-type on G. □

Let K be a field and let d be a discrete valuation on K. A well-known construction associates to this data a simplicial tree T. We require several facts about the action of $SL(2,K)$ on T and pause briefly to recall its definition (for additional information and details we refer to [27] or [7]). A vertex of T is by definition a homothety class of $\mathcal{O}$-lattices in the vector space $K \times K$ (two $\mathcal{O}$-lattices L and L' are homothetic if there exists $x \in K^\times$ such that $xL = L'$). Two vertices are adjacent if there are representative lattices for which $\pi L' \subseteq L \subseteq L'$. In the tree T there is a distinguished vertex, namely the class of the lattice $L = \mathcal{O} \times \mathcal{O}$. It is (the unique vertex) fixed by $SL(2,\mathcal{O})$.

5.5 Lemma. *Let* K *be a field with discrete valuation* d*. Let* T *be the associated simplicial tree* T *and* ν_0 *its distinguished vertex. If* $g = [g_{ab}] \in SL(2, K)$*, then*

$$\mathrm{distance}_T(\nu_0, g \cdot \nu_0) = 2 \max_{a,b} -\frac{\log d(g_{ab})}{\log d(\pi)}.$$

Proof. Both sides of the formula are both left and right $SL(2, \mathcal{O})$-invariant, as functions of g, so it suffices to prove the formula for one element in each double $SL(2, \mathcal{O})$-coset. Now if $g \in SL(2, \mathcal{O})$ then there exist $h_1, h_2 \in SL(2, \mathcal{O})$ such that $h_1 g h_2$ is a diagonal matrix of the form $\left(\begin{smallmatrix} \pi^n & 0 \\ 0 & \pi^{-n} \end{smallmatrix}\right)$. This follows by a row and column reduction argument similar to the one employed in the proof of Lemma 4.5. It therefore suffices to show that

$$\mathrm{distance}_T(\nu_0, \left(\begin{smallmatrix} \pi^n & 0 \\ 0 & \pi^{-n} \end{smallmatrix}\right) \nu_0) = 2|n|.$$

For $k = 0, \ldots, 2n$ the lattices

$$L_k = \mathcal{O}\text{-span of } \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pi^{-k} \end{pmatrix} \right\}.$$

define distinct vertices in T and a geodesic path of length $2|n|$ from ν_0 to the vertex $\left(\begin{smallmatrix} \pi^n & 0 \\ 0 & \pi^{-n} \end{smallmatrix}\right) \cdot \nu_0$. □

We now apply Lemma 5.4. Let K be a field with an archimedean valuation d, obtained from an embedding $K \subseteq \mathbb{C}$. Using this embedding we embed $SL(2, K)$ into $SL(2, \mathbb{C})$. The group $SL(2, \mathbb{C})$ acts on 3-dimensional real hyperbolic space, as in Lemma 5.4.

5.6 Lemma. *Denote by* x_0 *the unique point in* $SL(2, \mathbb{C})/SU(2)$ *which is fixed by the subgroup* $SU(2)$*. If* $g = [g_{ab}] \in SL(2, \mathbb{C})$ *then*

$$\cosh\bigl(\mathrm{distance}(x_0, g \cdot x_0)\bigr) = \frac{1}{2} \mathrm{Trace}(g^* g) = \frac{1}{2} \sum_{a,b} |g_{ab}|^2.$$

Proof. All parts of the formula are left and right $SU(2)$-invariant, as functions of g, so it suffices to check the formula on positive diagonal matrices. But these constitute a one-parameter group which acts by translation along a geodesic passing through x_0. The formula follows (up to an overall constant factor which we eliminate by scaling the metric on the symmetric space). □

Thanks to Lemmas 5.3 through 5.6, if K is a field, and if K is equipped with many valuations, then the group $G = SL(2, K)$ is equipped with many negative-type functions, whose growth behaviour on G we can moreover estimate in terms of the growth behaviour of the given valuations on K. We can now prove Theorem 5.1 by following the argument used to prove Theorem 4.1.

Proof of Theorem 5.1. Let K be a finitely generated field, and let Γ be a finitely generated subgroup of $SL(2, K)$. (We reduced to this case earlier.)

Let R be the subring of K generated by the (finitely many) matrix entries of a finite generating set for Γ and observe that $\Gamma \subseteq SL(2, R)$. Let $\{d_j\}$ be an R-proper family of valuations on K, as in Definition 2.1. Associated to each valuation d_j there is a negative-type function δ_j on $SL(2, K)$; if d_j is a discrete valuation then δ_j is defined as in Lemma 5.3, whereas if d_j is an archimedean valuation then δ_j is defined as in Lemma 5.4.

Define a negative-type function δ on G by

$$\delta(g) = \sum_j \varepsilon_j \delta_j(g),$$

where $\{\varepsilon_j\}$ is a fixed sequence of positive real numbers decreasing at a rate sufficient to guarantee that the sum converges for every $g \in G$ (such a sequence exists because the individual δ_j satisfy $\delta_j(g_1 g_2) \leq \delta_j(g_1) + \delta_j(g_2)$, so that convergence for every $g \in G$ is guaranteed by convergence for elements of a (finite) generating set). The restriction of δ to G is proper. Indeed if $\delta(g) \leq C$ then

$$\delta_j(g) \leq \varepsilon_j C, \quad \text{for all } j, \tag{5.1}$$

and applying Lemmas 5.5 and 5.6 we see that the entries of g are therefore d_j-bounded, for every j (by some quantity depending on C and ε_j) . Considering the definition of discrete embeddability, it follows that the set of possible matrix entries of those $g \in G$ for which $\delta(g) \leq C$ is finite. Therefore the set of all $g \in G$ for which $\delta(g) < C$ is finite, as required. □

6 Exactness of Linear Groups

In this section we strengthen Theorem 4.1 by proving that every countable linear group is *exact* in the sense of C^*-algebra theory. As a consequence we prove that the Novikov conjecture holds for subgroups of almost connected Lie groups.

The exactness condition is closely related to uniform embeddability. Indeed, every exact group is uniformly embeddable (see [25], [14]) and at present there is no example of a uniformly embeddable group which is not exact. However, exactness has some advantages over uniform embeddability — for example the class of exact groups is closed under group extensions (see [22]), whereas closure under extensions for uniformly embeddable groups is not known at present (but see [10]).

Rather than give a detailed account of exactness we shall present just one of several equivalent formulations of exactness (see [25]). For a fuller treatment of the topic the reader is referred to [2] or [29].

6.1 Definition. A countable discrete group is *exact* if there exists a sequence of positive-definite kernels $\phi_n\colon \Gamma \times \Gamma \to \mathbb{R}$ with the following two properties:

(i) For every finite set $F \subset \Gamma$ and every $\varepsilon > 0$ there is an N such that

$$g_1^{-1}g_2 \in F \quad \Rightarrow \quad \phi_n(g_1, g_2) > 1 - \varepsilon, \quad \forall n > N.$$

(ii) For every n there is a finite set $F \subset \Gamma$ such that

$$\phi_n(g_1, g_2) \neq 0 \quad \Rightarrow \quad g_1^{-1}g_2 \in F.$$

6.2 Theorem. *Let K be a field. Every countable subgroup of $GL(n, K)$ is exact.*

Remark. It makes sense to consider the exactness of non-countable groups. The previous theorem holds for *any* subgroup, countable or not.

A countable discrete group is exact if and only if all of its finitely generated subgroups are exact ([29], [10]). Therefore it suffices to prove Theorem 6.2 for finitely generated subgroups of $GL(n, K)$. Having restricted our attention to finitely generated subgroups we may also assume that the field K is finitely generated. Therefore *we assume that K is a finitely generated field and that Γ is a finitely generated subgroup of* $GL(n, K)$.

The characterization of exactness we are using (which we are taking as the definition) is formally very similar to the characterization of uniform embeddability given in Proposition 3.8. It is therefore not surprising that the proof of Theorem 6.2 has much in common with that of Theorem 4.1. We begin by recalling the necessary facts from Section 4.

Let K be a field and let d be a discrete valuation on K. We decompose the group $G = GL(n, K)$ as a product

$$G = PH,$$

and fix, for each $g \in G$, a decomposition $g = ph$. We define a map $G \to P$ by $g = ph \mapsto p$; as explained in the proof of Proposition 4.7 this map is isometric. In Lemma 4.6 we constructed certain positive-definite functions $\phi_m \colon P \to \mathbb{R}$. These lift to P-invariant positive-definite kernels, which we extend to positive definite kernels on G using the map $G \to P$ above:

$$\phi_m(p_1h_1, p_2h_2) = \phi_m(p_1^{-1}p_2).$$

According to the properties of the ϕ_m described in Proposition 4.7, and the fact that the map $G \to P$ is isometric, the positive-definite kernels on $G = GL(n, K)$ so constructed have the following properties:

(i) For every $C > 0$ and $\varepsilon > 0$ there exists m such that

$$\ell_d(g_1^{-1}g_2) \leq C \quad \Rightarrow \quad |1 - \phi_m(g_1, g_2)| < \varepsilon.$$

(ii) For every m there exists $B > 0$ such that

$$\phi_m(g_1, g_2) \neq 0 \quad \Rightarrow \quad \ell_d(g_1^{-1}g_2) < B.$$

(In fact any sufficiently large m will work in (i).)

We proceed similarly in the case of an archimedean valuation d on K. The proof of Proposition 4.4 exhibits a sequence of positive-definite functions ϕ_m on the solvable group $P \subseteq GL(n, \mathbb{C})$ which we again convert to positive-definite kernels on $GL(n, K) \subseteq GL(n, \mathbb{C})$, using the fact that $GL(n, \mathbb{C})$ is the product of P and the compact group $U(n)$. We obtain positive-definite kernels on $GL(n, K)$ with same properties (i) and (ii) above.

Putting the two constructions together we obtain the following result:

6.3 Lemma. *Let $\{d_j\}$ be a sequence valuations on K, each either discrete or archimedean. Let $\{\varepsilon_j\}$ and $\{A_j\}$ be positive sequences. There exist positive-definite kernels ϕ_j on $GL(n, K)$ such that*

(i) *If $\ell_j(g_1^{-1}g_2) \leq A_j$, then $|1 - \phi_j(g_1, g_2)| < \varepsilon_j$.*

(ii) *For every j there exists a constant B_j such that if $\ell_j(g_1^{-1}g_2) > B_j$, then $\phi_j(g_1, g_2) = 0$.* □

We now construct new positive-definite kernels on finitely generated subgroups of $GL(n, K)$ by combining the positive-definite kernels associated to a sequence of valuations on K. To do so we need the following fact:

6.4 Lemma. *Let X be any set.*

(i) *The pointwise product $\phi_1(x_1, x_2)\phi_2(x_1, x_2)$ of two positive-definite kernels is again a positive-definite kernel.*

(ii) *Should it converge, the product $\prod_{j=1}^{\infty} \phi_j(x_1, x_2)$ of a countable family of positive-definite kernels is again positive-definite.*

Remark. The convergence hypothesis is that for every $x_1, x_2 \in X$ the finite products $\prod_{j=1}^{J} \phi_j(x_1, x_2)$ converge pointwise as $J \to \infty$. It is permissible that the limit be zero.

Proof. The first statement is proved in [11, Corollary 5.5]. The second statement follows from the first, since a pointwise limit of positive-definite kernels is positive-definite. □

Remark. Thanks to the first part of the lemma, we can square the positive-definite functions which appear in Lemma 6.3 and thereby assume that they have the additional property $\phi_j(g_1, g_2) \geq 0$. This we shall do without further comment below.

Proof of Theorem 6.2. Let K be a finitely generated field and let $\Gamma \subseteq GL(n, K)$ be a finitely generated subgroup. (We reduced to this case earlier.) Let $R \subseteq K$ be the ring generated by the coefficients of the matrices in Γ, and let $\{d_j\}$ be an R-proper family of valuations on K. Let F be a finite subset of Γ and let $\varepsilon > 0$. For each j, let A_j be a constant such that

$$g_1^{-1} g_2 \in F \quad \Rightarrow \quad \ell_j(g_1^{-1} g_2) \leq A_j,$$

and let $\{\varepsilon_j\}$ be a positive sequence such that $\prod_j (1 - \varepsilon_j) \geq 1 - \varepsilon$. Now let $\{\phi_j\}$ be a sequence of positive-definite kernels on $GL(n, K)$ with the properties described in Lemma 6.3. Form the product

$$\phi(g_1, g_2) = \prod_j \phi_j(g_1, g_2),$$

which in view of Lemma 6.4 is a positive-definite kernel on $GL(n, K)$. If $g_1^{-1} g_2 \in F$ then $\phi_j(g_1, g_2) > 1 - \varepsilon_j$, for all j, and therefore

$$g_1^{-1} g_2 \in F \quad \Rightarrow \quad \phi(g_1, g_2) > 1 - \varepsilon.$$

If $\phi(g_1, g_2) \neq 0$ then of course $\phi_j(g_1, g_2) \neq 0$, for all j, and from this it follows that $\ell_j(g_1^{-1} g_2) < B_j$, for all j, where the constants B_j are as in Lemma 6.3. As a result, it follows from the definition of the length functions and of an R-proper family that there is a finite set $F_B \subseteq \Gamma$ such that

$$\phi(g_1, g_2) \neq 0 \quad \Rightarrow \quad g_1^{-1} g_2 \in F_B.$$

Inspecting Definition 6.1, we see that we have proved the exactness of Γ. □

As an application of Theorem 6.2 we obtain the following counterpart to Kasparov's proof of the Novikov conjecture for discrete subgroups of Lie groups [20].

6.5 Theorem. *Every countable subgroup of an almost connected Lie group is exact, and therefore uniformly embeddable into Hilbert space. As a result, the Novikov conjecture holds for all countable subgroups of almost connected Lie groups.*

Proof. Assume first that Γ'' is a countable subgroup of a *connected* Lie group. Using the adjoint representation of G we see that there is an extension of groups

$$1 \to Z \to \Gamma'' \to \Gamma' \to 1$$

where Z is abelian (in fact central in Γ'') and Γ' is linear [16]. Since the class of exact groups is closed under extensions, and since both Z and Γ' are exact, it follows that Γ'' is exact.

In the general case, if Γ is a subgroup of an almost connected Lie group then there is an extension

$$1 \to \Gamma'' \to \Gamma \to F \to 1,$$

where F is a finite group and where Γ'' is a subgroup of a connected Lie group. Since F and Γ'' are exact it follows that Γ is exact too. □

7 An Application to Relative Eta Invariants

Atiyah, Patodi and Singer [4] introduced a real-valued invariant $\widetilde{\eta}_\rho(M)$ of an odd-dimensional, smooth, closed and oriented manifold M, equipped with a finite-dimensional unitary representation $\rho : \pi_1(M) \to U(k)$ of its fundamental group. Although this invariant is *not* homotopy invariant, the third author has shown [31], using the Novikov Conjecture for subgroups of $GL_n(\overline{\mathbb{Q}})$, that for homotopy equivalent manifolds M and M' the difference $\widetilde{\eta}_\rho(M') - \widetilde{\eta}_\rho(M)$ is a *rational* number. In this section we shall use the main result of this paper to improve this result.

7.1 Theorem. *Let M and M' be homotopy equivalent smooth, closed and oriented, odd-dimensional manifolds with fundamental group π and let $\rho : \pi \to U(k)$ be a finite-dimensional unitary representation. Let*

$$R = \{\, p \text{ a prime} : \rho[\pi] \text{ has an element of order } p \,\}.$$

There is a positive integer S, all of whose odd prime factors belong to R, such that $\widetilde{\eta}_\rho(M) - \widetilde{\eta}_\rho(M') \in \frac{1}{S}\mathbb{Z}$.

Remark. When R is empty, or if π is torsion-free, the third author has conjectured that $\widetilde{\eta}_\rho$ is a homotopy invariant [30]. If R is non-empty, then the "integrality" statement above is, in some sense, the best possible, aside from the special role of the prime 2.

Proof. The idea of the proof of Theorem 7.1 is as follows. Define Γ to be the linear group $\rho[\pi]$. We shall realize the invariant $\widetilde{\eta}_\rho(M)$ (in $\mathbb{R}$ modulo $\mathbb{Z}[\frac{1}{2}]$) as the image of $[M] \in KO_n(B\Gamma)[\frac{1}{2}]$ (the K-homology class determined by the signature operator of M) under a map

$$KO_n(B\Gamma)[\tfrac{1}{2}] \longrightarrow \mathbb{R} \,/\, \mathbb{Z}[\tfrac{1}{2}]. \tag{7.1}$$

We shall also construct a map

$$KO_n(B\Gamma)[\tfrac{1}{2R}] \longrightarrow K_n(C^*_{\mathrm{red}}(\Gamma))[\tfrac{1}{2R}] \tag{7.2}$$

with the following properties:

(i) Thanks to Theorem 4.1, the map is (split) injective.

(ii) The image of the K-homology class $[M] \in KO_n(B\Gamma)[\frac{1}{2R}]$ under this map is a homotopy invariant.

(Here and subsequently, if A is an abelian group then $A[\frac{1}{2R}]$ denotes the tensor product with the ring obtained from $\mathbb{Z}$ by inverting 2 and the elements of R.) After inverting R in (7.1) we see right away that if M and M' are homotopy equivalent then the relative eta-invariants $\widetilde{\eta}_\rho(M)$ and $\widetilde{\eta}_\rho(M')$ are equal in $\mathbb{R} \,/\, \mathbb{Z}[\frac{1}{2R}]$, as required.

The map (7.1) is constructed as follows. Let $\Omega_m(X)$ denote the m-dimensional, smooth, oriented bordism group of the space X (thus classes in $\Omega_m(X)$ are represented by maps $\phi : N \to X$ where N is a closed, oriented, m-dimensional, smooth manifold). The direct sum $\Omega_*(X) = \oplus_m \Omega_m(X)$ is a module over $\Omega_*(pt)$, which is itself a ring. The map which sends $[N] \in \Omega_{4k}(pt)$ to $\mathrm{Sign}(N)$, and which is zero on all components $\Omega_m(pt)$, where m is not divisible by 4, is a ring homomorphism from $\Omega_*(pt)$ to $\mathbb{Z}[\frac{1}{2}]$. Using it, we can form the tensor product

$$\Omega_*(X) \otimes_{\Omega_*(pt)} \mathbb{Z}[\tfrac{1}{2}].$$

The tensor product is naturally a $\mathbb{Z}/4\mathbb{Z}$-graded abelian group, and according to Sullivan's "Conner-Floyd theorem" [24] the signature operator provides an isomorphism

$$\Omega_*(X) \otimes_{\Omega_*(pt)} \mathbb{Z}[\tfrac{1}{2}] \cong KO_*(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \subseteq K_*(X) \otimes \mathbb{Z}[\tfrac{1}{2}]. \tag{7.3}$$

Now let $X = B\Gamma$. According to the APS index theorem [3], the relative eta-invariant $\widetilde{\eta}_\rho$ defines a homomorphism

$$\Omega_*(B\Gamma) \otimes_{\Omega_*(pt)} \mathbb{Z}[\tfrac{1}{2}] \to \mathbb{R}/\mathbb{Z}[\tfrac{1}{2}]. \tag{7.4}$$

This is because if $[N] = [N']$ in the left hand side then there is a compact manifold mapping to $B\Gamma$ whose boundary is the disjoint union of 2^a copies of N, the same number of copies of $-N'$ and product manifolds $A_i \times B_i$ with $\mathrm{Sign}(A_i) = 0$. Since $\widetilde{\eta}_\rho(A \times M) = \mathrm{Sign}(A) \cdot \widetilde{\eta}_\rho(M)$ (see [12]) the product manifolds have trivial relative eta-invariant, and since the relative eta-invariants of all the boundary components add up to an integer (by the APS index theorem) we see that

$$\widetilde{\eta}_\rho(N) = \widetilde{\eta}_\rho(N') \in \mathbb{R}/\mathbb{Z}[\tfrac{1}{2}],$$

as required. Putting together (7.3) and (7.4) we obtain the map (7.1) that we need.

It remains to define the map (7.2). There is a natural map

$$KO_n(B\Gamma)[\tfrac{1}{2}] \longrightarrow K_n(B\Gamma)[\tfrac{1}{2}] \tag{7.5}$$

which is split injective.[2] Now the left-hand side of the Baum-Connes assembly map (0.1) (in the case of trivial coefficient C^*-algebra $A = \mathbb{C}$) is the Kasparov equivariant K-homology of the classifying space $\mathcal{E}\Gamma$ for proper Γ-actions (we shall denote this by $K_n^\Gamma(\mathcal{E}\Gamma)$). See [5]. There is a natural map

$$K_n(B\Gamma) \longrightarrow K_n^\Gamma(\mathcal{E}\Gamma) \tag{7.6}$$

[2] We already invoked this when we associated classes in $KO_n(B\Gamma)[\frac{1}{2}]$ to elliptic operators.

and after inverting R this map becomes a split-injection. Indeed if $\mathcal{B}\Gamma$ is the quotient of $\mathcal{E}\Gamma$ by Γ then there is a map from $K_n^\Gamma(\mathcal{E}\Gamma)$ to $K_n(\mathcal{B}\Gamma)$ for which the composition

$$K_n(B\Gamma) \to K_n^\Gamma(\mathcal{E}\Gamma) \to K_n(\mathcal{B}\Gamma)$$

is induced by the natural map from $B\Gamma$ to $\mathcal{B}\Gamma$. Standard arguments show that the induced map is an isomorphism after inverting the primes in R (compare Lemma 2.8 in [23]). Putting together (7.5) and (7.6) we obtain the split injection (7.2). The fact that the class $[M]$ is homotopy invariant in the image follows from the homotopy invariance of the K-theoretic index of the signature operator (see [5] again for references). □

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