Homework for students attending Monique Chyba and Bernard Bonnard courses

Exercice 1

We consider the minimization problem in $\mathbb{R}$,with fixed extremities :

$$Min \int_{t_0}^{t_1} L(t, x(t), \frac{dx}{dt}))dt.$$

Let $t \to x(t)$ be a piecewise smooth minimizer, with an angular point at $c\epsilon]t_0, t_1[$. Use the fundamental formula to get necessary optimality conditions at c. Present those conditions using Hamiltonian formalism. Compare with the conditions of the maximum principle : continuity of the adjoint vector and constant Hamiltonian.

Reference Erdmann and Weierstrass conditions (1877), Bolza book : calculus of variations, Chelsea

Problem 1

We consider the following SR-geometry problem associated to the constraints :

$$\frac{dq(t)}{dt}\epsilon D(q(t), q = (x, y, z), D = ker\alpha, \alpha = dz - \frac{y^2}{2}dx.$$

Let $F_1 = \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}, F_2 = \frac{\partial}{\partial y}$ and introduce $F_3 = \frac{\partial}{\partial z}$ and consider the SR problem (Martinet SR-flat case)

$$\frac{dq}{dt} = u_1 F_1 + u_2 F_2, Min \int_0^T (u_1^2 + u_2^2)dt.$$

1. Compute the abnormal geodesic curves
2. Set $H_1 =< p, F_i(q) >, i = 1, 2, 3$. Compute the geodesic equations in the normal case using the coordinates (x, y, z, H_1, H_2, H_3).Integrate the geodesic equations initiating from $q(0) = 0$ using elliptic functions (reference book Lawden, elliptic function, Sringer Verlag.).

Problem 2

Consider a rigid spacecraft controlled by gas jets and described by the following system

$$\frac{dR(t)}{dt} = S(\omega(t)R(t)$$

$$\frac{d\omega}{dt}(t) = Q(\omega(t)) + u(t)b,$$

wher $R\epsilon SO(3)$ = group of rotations in dimension 3 with determinant 1, $S(\omega)$ is the antisymmetric matrix $\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$ and Q is the vector field

$$Q = a_1\omega_2\omega_3\frac{\partial}{\partial\omega_1} + a_2\omega_1\omega_3\frac{\partial}{\partial\omega_2} + a_3\omega_2\omega_1\frac{\partial}{\partial\omega_3}, a_1 > 0, a_2 < 0, a_3 > 0.$$

The control $u(t)$ is a piecewise constant mapping valued in $[-1, +1]$ and the vector $b = (b_1, b_2, b_3)$ described the position of the control device on the satellite.

1. Show that the free motion corresponding to a control identically zero is Poisson stable (Hint one can use Poinsot interpretation of the motion).

2. Prove that the system is controlablle if and only the control torque represented by b does not belong to one of the axis $0\omega_i$ or one of the two planes given by the equations : $a_3\omega_1^2 - a_1\omega_3^2 = 0$. (interpret these two planes with respect to the properties of the free motion).