Lecture 2 Riemannian and Sub-Riemannian geometry

Definition

- Riemannian geometry

$$\frac{dx}{dt} = \sum_{i=1,,n} u_i F_i(x), Min u \to \int_0^T \sum u_i^2(t) dt$$

- SR-geometry

$$\frac{dx}{dt} = \sum_{i=1,,m} u_i F_i(x), Min u \to \int_0^T \sum u_i^2(t) dt, m < n$$

where n is the dimension of the ambient manifold.

Concepts of Symplectic geometry

- **H** hamiltonian vector field associated to the Hamilton function H : $\frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p}$
- Poisson bracket of two hamitonians H, G : $\{H, G\} = dH(\mathbf{G}) = \frac{\partial H}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial x}\frac{\partial H}{\partial p}$ (Eisnstein convention)
- If F, G vector fields with symplectif lifts $H_F =< p, F(x) >$, $H_G =< p, G(x) >$ then $\{H_F, H_G\}(z) =< p, [F, G](x) >$, with $z = (x, p)$ Darboux coordinates.

Computations of geodesics equations in Poincaré coordinates

Introduce $H = (H_1, .., H_n)$ where H_i is the symplectic lift of the vector filed F_i. Replace $(x, p) \to (x, H)$.

Then

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \sum u_i F_i = \sum H_i F_i$$

$$\frac{dH_i}{dt} = dH_i(\mathbf{1/2} \sum \mathbf{H_i^2}) = \sum \{H_i, H_j\} H_j$$

and

$$\{H_I, H_j\} =< p, [F_i, F_j](x) >, [F_i, F_j](x) = \sum c_{ij}^k(x) F_k(x)$$

Note invariant case : the structure coefficients c_{ij}^k are constant.

Example Flat case

$$[F_i, F_j](x) = 0.$$

Hence

$$\frac{dH_i}{dt} = 0, H_i = c_i.$$

Frobenius : Choose a ccordinates system such that $F_i = \frac{\partial}{\partial x_i}$.
One gets : the solutions of geodesics equations are lines.

Basic facts about SR-geometry

Introduce the distribution $D = span\{F_1, F_2, ..., F_m\}$ asociated to the (in general) non holonomic constraints $\frac{dx}{dt} \epsilon D(x(t))$.One can complete the distribution to form a frame $\{F_1, ..., F_m, ..., F_n\}$ (Geometry : the SR metric is defined as the restriction to a Riemannian metric to the distribution).

First step compute the geodesics equations using the weak maximum principle

One has two cases called respectively the normal case and the abnormal case

Normal case in SR-geometry

It corresponds to a singularity to the cost extended system (similar to the Riemannian case) with pseudo -Hamiltonian

$$H(z, u) = \sum_{i=1,...,m} u_i H_i - 1/2 \sum_{i=1,...,m} u_i^2$$

which gives using tthe weak maximum principle the true Hamiltonian

$$H = \frac{1}{2} \sum_{i=1,...,m} H_i^2.$$

Abnormal case Example $m = 2$.

They correspond to the singularities of the system only and are associated to the pseudo-Hamilonian

$$H = \sum_{i=1,.2} u_i H_i$$

and from the maximum principle one must have identically

$$H_1(z(t)) = H_2(z(t)) = 0.$$

Differentiate twice along the solutions one gets the relations

$$\{H_1, H_2\}(z(t)) = u_1\{\{H_1, H_2\}, H_1\}(z(t)) + u_2\{\{H_1, H_2\}, H_2\}(z(t)) = 0.$$

The second relation allows to compute the control (parameterized by arc-length) : $u_1^2 + u_2^2 = 1$.

Usual definitions and concepts in SR-geoemtry similar to the Riemannian case

Notation $H_n = 1/2 \sum_{i=1,\ldots,m} H_i^2$: Hamiltonian in the normal case .

Solution $z(t, z_0) = expt\mathbf{H}_n(z)$. : normal extremal , geodesic : projection on the ambient manifold.

Exponential mapping : fix $x(0) = x_0$, **and let** $\pi : (x, p) \to x$. $exp_{x_0}(p(0), t) \to \pi(z(t, z_0))$.

One can parametrerized by arc-length : impose $H_n = 1/2$

Sub-Riemannian distance

Length of the minimizing geodesic joining x_0 to x_1. (Chow theorem ensures the existence of the distance and Filipov theorem ensures exitence oof minimizers).

Ball and sphere with radius r

$$B(x_{x_0}, r) = \{x, d(x_0, x) \le r\}, S(x_0, r) = \{x, d(x_0, x) = r\}.$$

Ultimate goal : compute the sphere

In SR-geometry the computation is intricated even form small radius : singularities

The concept of conjugate point

It is a basic concept in SR-geometry : the definition is similar to the Riemannian case.

Definition

Let $t \to x(t, p(0))$ be a reference geodesic. A conjugate time is a time t_c such that the exponential is not of full rank (not immersive).

Proposition

If $t > t_{1c}$ (First conjugate time) the geodesic is no more minimizing for the C^1-topology on the set of curves.

Notation

$C(x_0)$: conjugate locus=set of first conjugate points for geodesics emanating from x_0. Cut locus $C_\Sigma(x_0)$ = set of points along geodesics where the global optimality is lost.

Classification of distributions

An important step in the analysis of SR)-geometry is the problem of classification of sdistributions which goes back to Cartan, Goursat, Darboux and was well studiedmore recently in the framework of singularity theory (Martinet, Zhitomirski). Note that this classification is related to the abnormal curves associated to the distribution(contribution to control theory to the computations of invariants).

Definition

A discrete set of invariants is simply the growth f vector at a point : let $D_1 = D, D_2 = [D, D_1]$, ...,$D_k = [D_{k-1}, D_1]$ and $(n_1, n_2, ..., n_k, ...)$ the respective dimensions. (Use Chow theorem to end the sequence)

Classification of two dimensional distributions in dimension 3 at a point x_0 (stable cases)

Contact case

- $D = span\{F_1, F_2\}$
- $[F_1, F_2](q_0) \notin D(q_0)$

Normal form

Coordinates $q = (x, y, z), D = ker\alpha, \alpha = ydx + dz$.

Observe : $d\alpha = dy \wedge dx$: Darboux form, $\frac{\partial}{\partial z}$ = Lie bracket $[F_1, F_2]$: characteristic direction

Martinet case

- $[F_1, F_2](q_0)\epsilon D(q_0)$
- $[[F_1, F_2], F_1](q_0]$ or $[[F_1, F_2], F_2](q_0] \notin D(q_0)$.

Normal form

$D = ker\alpha$, $\alpha = dz - y^2/2dx$.

Interpretation : Martinet surface $M = det(F_1, F_2, [F_1, F_2]) = 0$ identified to $y = 0$. Abnormal curves in this surface identified to the vector field $\frac{\partial}{\partial x}$

Remark

In both cases the distributions are nilpotent

Working example (Brockett-birth of SR-geometry) The Heisenberg case

Model

$q = (x, y, z)$ local coordinates near 0

$$\frac{dq}{dt} = u_1F_1 + u_2F_2, F_1 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial x}, F_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial x},$$

Note : Contact case at 0. The SR-metric is given by $Min \int_0^T (u_1^2 + u_2^2)dt$. Amount to minimize the euclidian length of the projection curve on the plane (x, y).

Relation with Dido problem

one has $z_1 - z_0 = \int_0^T (\frac{dx}{dt}y - \frac{dy}{dt}x)dt$: integrant proportional to the aera swept by the curve in the plane (x, y). Dual to the problem : among the closed curves in the plane whose length is fixed find the one with maximum enclosed aera.

SOLUTIONS : CIRCLES.

Computations : only normal curves (Contact case)

Poincaré coordinates introducing $F_3 = [F_1, F_2] = \frac{\partial}{\partial z}$.

$$\frac{dH_1}{dt} = \{H_1, H_2\}H_2 = H_2H_3, \frac{dH_2}{dt} = -H_1H_3, \frac{dH_3}{dt} = 0.$$

Integration

$H_3 = \lambda$ (constant), $H_1 = Asin(\lambda t + \varphi)$. (Linear pendulum).

Geodesics starting from 0

Deduce geodesics

- $\lambda = 0$: Straigth lines in the plane (x, y)
- λ non zero (can be taken positive)

$$x(t) = \frac{A}{\lambda}[sin(\lambda t + \varphi) - sin\varphi], y(t) = \frac{A}{\lambda}[cos(\lambda t + \varphi) - cos\varphi]$$

Circles in the palne (x, y)

$$z(t) = \frac{A^2}{\lambda}t - \frac{A^2}{\lambda^2}sin(\lambda t).$$

Note : λ, A, φ are the constant of integrations

Non trivial exercice

Check that the first conjugate time occurs at $t_{1c} = 2\pi/\lambda$ for a non planar geodesics and correspond also to the cut time.

Computation

exponentiel mapping (λ, A, φ)not of full rank. Computations are possible without numerics since we have the explicit form of the solutions using elementary functions.

Conclusion Represent the sphere of radius r and show it not smooth when intersecting the z-axis (conjugate point computation)

Lecture 3 An application of SR-geometry : the swimming problem at low Reynolds number

They are many applications of SR-geometry, e;g. to planning motions in robotics butr we concentrate to a recent one associated to locomation. It is the problem of swimming for mico-organisms called the swimming problem at low Reynolds see the introductory paper by Purcell : life at lowReynolds number, American J. Physics, Vol 5, No 1, Jan 1977.

Matthematical framework

The first result is the so-called scallop theorem

Theorem A scallop cannot swim

The model is

$$\frac{dx}{dt} = \left[\frac{sin\alpha}{2 - cos^2\alpha}\right]\frac{d\theta}{dt}$$

where x is ths displacement and θ is the shape variable which goes from $\pi/2 - \gamma \to \pi/2$ when opening and $\pi/2 \to \pi/2 - \gamma$ when closing producing a total displacement of zero (reversibility) obvious from the form of the equation.

Conclusion at least two shape variables (links) are necessary to swim.

This leads to the famous Purcell swimmer. It is formed by to two links to change the shape with the dynamics $\frac{d\theta_i}{dt} = u_i$ and the position of the body of the swimmer is given by $q = (x, y, \alpha)$and the dynamics is described by $\frac{dq}{dt} = \sum u_i F_i(\theta, \alpha)$.

The deformation in the shape variables produces a displacement which is a very nice consequence of Baker-Campbell-Hausdorff formulae. Applying the sequence $u_1 = 1, u_2 = -1, u_1 = -1, u_2 = 1$ leads to compute the composition

$$exptF_2oexptF_1oexptF_2exp - tF_1 = exp(t^2[F_1, F_2] + o(t^2)).$$

For small t it amounts to a displacement of $t^2[F_1, F_2]$ evaluated at the point q

NB Compare with Becker, Koehler, Store, Journal of Fluid dynamics , 2003, vol 490, 15-35.

Optimal control

The underlying optimal control problem is well known : from fluid mechanics : it is the mechanical power dissipated by the fluid viscous drag of forces and torques resisting to the motion of the links.

Hence the Purcell problem is of the form

$$\frac{dq}{dt} = D(\alpha)G(\theta)\frac{d\theta}{dt}$$

where G is a very complicated matrix and the cost is of the form $u \to Min \int_0^{2\pi} L(\theta, \frac{d\theta}{dt})dt$ given by the mechanical power expended (quadratic form in $\frac{d\theta}{dt}$).

Simplified model the copepod swimmer Daisuke Takagi

Small sphere with small radius with n pairs of symmetric links with equal length and the swimmer velocity is

$$\frac{dx}{dt} = \frac{\sum \frac{d\theta_i}{dt} sin\theta_i}{\sum 1 + sin^2\theta_i}.$$

This model is motiavated by

- model for a abundant variety of zooplankton called the copepod
- Purcell swimmer is very complicated : five dimensional model and the approximating Lie algebra is the Cartan nilpotent model.

General concept

The concept of stroke is the following. It is a periodic motion in the shape space (the period can be normalized to 2π). The optimal control problem is to compute the optimal strokes.

Note

The weak maximum principle has to be applied with an additional transversality conditions related to **periodic** strokes. The adjoint vector p_θ dual to the shape variables has to satisfy $p_\theta(0) = p_\theta(2\pi)$.

Conclusion

The copepod swimmer is analyzed in full details in the paper : A note about the geometric optimal control of teh copepod swimmer : various strokes are geometrically discribed and numerically computed in the optimal control framawork : non sel intersecting , eight, limacon and their optimality discussed using the concept of conjugate points and computing the energy function.