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ENERGY-MINIMAL TRANSFERS IN THE VICINITY OF THE LAGRANGIAN POINT L_1.

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Abstract. This article deals with the problem of computing energy-minimal trajectories between the invariant manifolds in the neighborhood of the equilibrium point L_1 of the restricted 3-body problem. Initializing a simple shooting method with solutions of the corresponding linear optimal control problem, we numerically compute energy-minimal extremals from the Pontryagin's Maximum principle, whose optimality is ensured thanks to the second order optimality condition.

1. **Introduction.** Computing energy-minimal orbit transfers is one of the crucial challenges to take up to design space missions using low-propulsion. In particular, the SMART-1 mission from the European Space Agency, see [11], has motivated numerous studies dealing with low thrust trajectories from the Earth to the Moon, using for instance simple feedback laws [3] or the transcription method [2]. In [4], optimal transfers between quasi-Keplerian orbits in the Earth-Moon system are computed basing on numeric methods connected with fundamental results from geometric control theory. The free motion of the spacecraft is described by the equations of the planar restricted 3-body problem, [13]: two primaries are circularly revolving around their center of mass with a constant angular velocity under the influence of their mutual gravitational attraction and a third body with negligible mass is moving in the plane defined by the motion of the two primaries. Adding control terms in the equations of motion, extremal curves solutions of the Pontryagin's Maximum Principle [10] are numerically computed using simple shooting and smooth continuation methods. Their local optimality is checked according with the second order optimality condition related with the notion of conjugate points. In figure 1 is displayed an example of such an optimal trajectory.

In this article, the above methods are used to compute energy-minimal transfers in the vicinity of the equilibrium point L_1 of the spatial restricted 3-body problem where the vertical dimension is taken into account. Indeed, the study of the flow in the equilibrium region exhibits invariant manifolds of orbits asymptotic to an invariant 3-sphere of bounded orbits, see [9]. These invariant manifolds can be used as low energy passageways connecting primaries' attraction areas. This is why we investigate the problem of minimizing the energy cost to reach one from one other.

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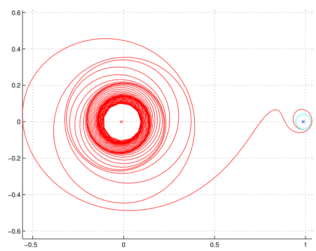


FIGURE 1. Energy minimal Earth-Moon trajectory in the rotating frame. The neighborhood of the point L_1 acts as the way from Earth's to Moon's gravity area.

The first section of this article focus on the dynamics of the spatial problem in the vicinity of the point L_1. Equations of motion can be written in Hamiltonian form and the study of the linearized system shows that local behavior is of the type saddle×center×center, allowing to classify orbits in bounded, asymptotic, transit and non-transit orbits, see [9]. The second section is devoted to the linearized control system. Using standard results from the linear control theory, see [8], we explicitly compute the optimal control used to reach the instable manifold toward the Moon from the stable manifold from the Earth. The initial adjoint vector η_0 from the maximum principle depends on the initial condition in the phase space z_0 and can be computed integrating backwards a Riccati matricial equation. In the last section, we use this η_0 to initialize the simple shooting method and numerically compute energy-minimal extremal solutions associated with the non- linear control system, whose local optimality is ensured by the second order conditions.

2. **Spatial problem and dynamics in the vicinity of equilibrium points.** Let us recall the equations of the three degree-freedom circular restricted 3-body problem, see [13] for further details. Units of time, mass and length are normalized so that the sum of the primaries masses, the distance between the primaries, their angular velocity and the gravitational constant are 1. Choosing a positively oriented synodic reference system$\{O, X, Y, Z\}$, the origine is the barycenter of the two primaries. The biggest primary (the Earth) with mass $1-\mu$ is located at $(-\mu, 0, 0)$ and the smallest one (the Moon) with mass mass μ is located at $(1-\mu, 0, 0)$. The equations of motion of the third body (the spacecraft) take the form

$$\ddot{X} - 2\dot{Y} = \frac{\partial V}{\partial X},\ \ddot{Y} + 2\dot{X} = \frac{\partial V}{\partial Y},\ \ddot{Z} = \frac{\partial V}{\partial Z} \tag{1}$$

where the potential is given by

$$V = \frac{1}{2}(X^2+Y^2) + \frac{1-\mu}{((X+\mu)^2+Y^2+Z^2)^{\frac{1}{2}}} + \frac{\mu}{((X-1+\mu)^2+Y^2+Z^2)^{\frac{1}{2}}}.$$

Setting $p_X = \dot{X} - Y$, $p_Y = \dot{Y} + X$ and $p_Z = \dot{Z}$, one can write the equations in Hamiltonian form associated with the Hamiltonian function

$$H = \frac{1}{2}(p_X^2 + p_Y^2 + p_Z^2) + Y p_X - X p_Y - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}. \tag{2}$$

where ρ_1 and ρ_2 are the distances between the spacecraft and the primaries. The equilibrium points of the problem are well known. They all belong to the (X,Y)-plane and split in two different types. Firstly, the collinear points L_1, L_2 and L_3

are located on the line $y = 0$ defined by the primaries. Secondly, the equilateral points L_4 and L_5 form with the two primaries equilateral triangles. To study the dynamics in the vicinity of a collinear equilibrium point $L_{i=1,\ 2,\ 3}$, one translates the origine to the location of L_i and applies some scaling so that the distance γ_j from L_i to the closest primary is 1. The equations of motion become

$$\begin{aligned} \ddot{x} - 2\dot{y} - (1 + 2c_2)x &= \frac{\partial}{\partial x} \sum_{n\geq 3} c_n \rho^n P_n(\frac{x}{\rho}), \\ \ddot{y} + 2\dot{x} + (c_2 - 1)y &= \frac{\partial}{\partial y} \sum_{n\geq 3} c_n \rho^n P_n(\frac{x}{\rho}) \\ \ddot{z} + c_2 z &= \frac{\partial}{\partial z} \sum_{n\geq 3} c_n \rho^n P_n(\frac{x}{\rho}) \end{aligned} \tag{3}$$

where P_n denotes the Legendre polynomial of order n, $\rho = x^2 + y^2 + z^2$ and the coefficients c_n depend on both libration points and the constant μ. Skipping the non-linear terms, one obtains the linearized equation

$$\ddot{x} - 2\dot{y} - (1 + 2c_2)x = 0,\ \ \ddot{y} + 2\dot{x} + (c_2 - 1)y = 0, \ddot{z} + c_2 z = 0. \tag{4}$$

Defining p_x, p_y and p_z as previously, the linearized equation is equivalent to the Hamiltonian system associated with the function

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y - \frac{c_2}{2}(2x^2 - y^2 - z^2). \tag{5}$$

It is not difficult to check that the linear behavior in the vicinity of $L_{i=1,\ 2,\ 3}$ is of type saddle×center×center with two real and four imaginary eigenvalues denoted $(\pm\lambda_1, \pm i\omega_1, \pm i\omega_2)$ see [9]. Moreover, one can show that the matrix

$$C = \begin{pmatrix} \frac{2\lambda_1}{s_1} & 0 & 0 & -\frac{2\lambda_1}{s_1} & \frac{2\omega_1}{s_2} & 0 \\ \frac{\lambda_1^2 - 2c2 - 1}{s1} & \frac{-\omega_1^2 - 2c2 - 1}{s2} & 0 & \frac{\lambda_1^2 - 2c2 - 1}{s1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\omega_2}} & 0 & 0 & 0 \\ \frac{\lambda_1^2 + 2c2 + 1}{s1} & \frac{-\omega_1^2 + 2c2 + 1}{s2} & 0 & \frac{\lambda_1^2 + 2c2 + 1}{s1} & 0 & 0 \\ \frac{\lambda_1^3 + (1 - 2c2)\lambda_1}{s1} & 0 & 0 & \frac{-\lambda_1^3 - (1 - c2)\lambda_1}{s1} & \frac{-\omega_1^3 + (1 - 2c2)\omega_1}{s2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\omega_2} \end{pmatrix}$$

where

$$s_1 = (2\lambda_1((4 + 3c_2)\lambda_1^2 + 4 + 5c_2 - 6c_2^2))^{\frac{1}{2}},\ \ s_2 = (\omega_1((4 + 3c_2)\omega_1^2 - 4 - 5c_2 + 6c_2^2))^{\frac{1}{2}},$$

defines a symplectic change of variables that, denoting $v = (q_1, q_2, q_3, p_1, p_2, p_3)$ the new coordinates, casts the Hamiltonian 5 into

$$H = \lambda_1 q_1 p_1 + \frac{\omega_1}{2}(q_2^2 + p_2^2) + \frac{\omega_2}{2}(q_3^2 + p_3^2). \tag{6}$$

The analysis of the flow of the linearized system in the equilibrium region $\mathcal{R}$, determined for positive h and c by $H = h$ and $|q_1 - p_1| \leq c$, can be performed noting that, when $q_1 p_1 \neq 0$, the projections of the orbits in the (q_1, p_1)-plane lie on the branches of hyperbolas $q_1 p_1$=constant, since $q_1 p_1$ is a first integral of the linearized equation. One thus distinguishes four categories of orbits [9]:

- the point $q_1 = p_1 = 0$ corresponds to an invariant 3-sphere S_h^3 of *bounded orbits*,

- the *asymptotic orbits* gather the stable and unstable manifolds of S_h^3, respectively denoted $W_{\pm}^s(S_h^3)$ and $W_{\pm}^u(S_h^3)$ and given by

$$
\begin{aligned}
W_{\pm}^s(S_h^3) &= \{\frac{\omega_1}{2}(q_2^2+p_2^2)+\frac{\omega_2}{2}(q_3^2+p_3^2)=h,\ q_1=0,\ p_1 \gtreqless 0\} \\
W_{\pm}^u(S_h^3) &= \{\frac{\omega_1}{2}(q_2^2+p_2^2)+\frac{\omega_2}{2}(q_3^2+p_3^2)=h,\ p_1=0,\ q_1 \gtreqless 0\},
\end{aligned}
\tag{7}
$$

- the hyperbolic segments $q_1p_1 = \text{constant} > 0$ correspond to *transit orbits*,
- the hyperbolic segments $q_1p_1 = \text{constant} < 0$ correspond to *non-transit orbits*.

Let us mention that the non-linear dynamics in the Region $\mathcal{R}$ is qualitatively the same that the linear one, see [9]. In this case, there exists a normally perodic invariant manifold $\mathcal{M}_3^h$ which still has stable and instable manifolds which can be approximated by the invariant manifolds $W_{\pm}^s(S_h^3)$ and $W_{\pm}^u(S_h^3)$, the non-linear terms being much smaller than the linear ones in a neighborhood of the collinear point.

3. **Linear control system with energy cost around the point L_1.** From now on, we focus on studying the controlled dynamics around the Lagrangian point L_1. Our aim is to compute energy-minimal transfers reaching the positive branch of the instable manifolds $W_+^u(S_h^3)$ from the positive branch of the stable manifolds $W_+^s(S_h^3)$. Indeed, $W_+^s(S_h^3)$ (resp. $W_+^u(S_h^3)$) is a dynamical channel which connects the Earth's (resp. Moon's) attraction area and a closed neighborhood of L_1 as time increases (resp. decreases) that justifies the interest of such transfers for designing low-energy Earth-Moon trajectories. Let us start by investigating the linearized control system

$$\ddot{x} - 2\dot{y} - (1+2c_2)x = u_1,\ \ddot{y} + 2\dot{x} + (c_2-1)y = u_2,\ \ddot{z} + c_2 z = u_3. \tag{8}$$

which is obtained adding control terms in the equations of motion 4. Thus, the Hamiltonian function 5 becomes

$$H = \frac{1}{2}(p_x^2+p_y^2+p_z^2) + yp_x - xp_y + c_2(x^2+\frac{y}{2}+\frac{z}{2}) - u_1x - u_2y - u_3z. \tag{9}$$

Applying the symplectic change of variable C, equation 8 can be written

$$\dot{v} = Av + Bu \tag{10}$$

where

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_2 \\ 0 & 0 & 0 & -\lambda_1 & 0 & 0 \\ 0 & -\omega_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_2 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} \frac{2\lambda_1}{s_1} & -\frac{\lambda_1^2-2c2-1}{s1} & 0 \\ -\frac{2\omega_1}{s_2} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2\lambda_1}{s_1} & \frac{\lambda_1^2-2c2-1}{s1} & 0 \\ 0 & \frac{-\omega_1^2-2c2-1}{s2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\omega_2}} \end{pmatrix}.$$

Considering no constraints on the control bound, the Kalman condition, see [8], asserts that the system 10 is controlable. Let us fix a transfer time t_f. Therefore, determining an energy-minimal transfer from $W_+^s(S_h^3)$ to $W_+^u(S_h^3)$ consists in solving the following linear optimal control problem with quadratic cost

$$
\begin{gathered}
\dot{v} = Av + Bu \\
\min_{u(.)\in\mathbb{R}^2} \int_0^{t_f} u_1^2 + u_2^2 + u_3^2 dt \\
v(0) \in W_+^s(S_h^3),\ \ v(t_f) \in W_+^u(S_h^3).
\end{gathered}
\tag{11}
$$

Set $v_0 \in W^s_+(S^3_h)$. From the standard linear quadratic maximum principle, see [8], there exists an unique optimal control u^* solving 11; the corresponding optimal response v^* and adjoint vector η^* are found as any solutions of the system

$$\begin{gathered} \dot{v} = Av + BB^T\eta,\ \dot{\eta} = -A^T\eta \\ v(0) = v_0,\ v(t_f) \in \partial W^u_+(S^3_h) \\ \eta(t_f) \text{ is interior normal to } W^u_+(S^3_h) \text{ at } v(t_f) \end{gathered} \tag{12}$$

and u^* is given by

$$u^*(t) = B^T\eta(t). \tag{13}$$

Given an initial $\eta_0 \in T^*_{v_0}W^s_+(S^3_h)$ such that (v_f, η_f) fullfils the final transversality condition of the system 12, the solution of the linear differential system 12 is

$$\begin{pmatrix} v(t) \\ \eta(t) \end{pmatrix} = \exp^{t\begin{pmatrix} A & BB^T \\ 0 & -A^T \end{pmatrix}} \begin{pmatrix} v_0 \\ \eta_0 \end{pmatrix} \tag{14}$$

and substituting in 13, the optimal control can be explicitly written

$$u^*(t) = \begin{pmatrix} \frac{2\lambda_1}{s_1}(\eta^0_{q_1}e^{-\lambda_1 t} + \eta^0_{p_1}e^{\lambda_1 t}) - \frac{2\omega_1}{s_2}(\eta^0_{q_2}\cos(\omega_1 t) + \eta^0_{p_2}\sin(\omega_1 t)) \\ \frac{\lambda_1^2 - 2c2 - 1}{s1}(-\eta^0_{q_1}e^{-\lambda_1 t} + \eta^0_{p_1}e^{\lambda_1 t}) - \frac{\omega_1^2 + 2c2 + 1}{s2}(\eta^0_{p_2}\cos(\omega_1 t) - \eta^0_{q_2}\sin(\omega_1 t)) \\ \frac{1}{\sqrt{\omega_2}}(\eta^0_{p_3}\cos(\omega_2 t) - \eta^0_{q_3}\sin(\omega_2 t)) \end{pmatrix}.$$

Such an initial adjoint vector η_0 can be determined integrating backwards a Riccati matricial equation, see [8]. Indeed the condition $\eta(t_f)$ is interior normal to $W^u_+(S^3_h)$ at $v(t_f)$ involves

$$\eta^{*T}(t_f) = -v^{*T}(t_f)Q \tag{15}$$

where the matrix Q is given by

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k\omega_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & k\omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & k\omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & k\omega_2 \end{pmatrix}$$

and the constant k is stictly positive. Since the couple (v^*, η^*) satisfies

$$\dot{v} = Az + BB^T\eta,\ \dot{\eta} = -A^T\eta,$$

it comes that $\eta^{*T}(t) = v^{*T}(t)E(t)$ where E is the solution of the Riccati matricial equation

$$\begin{gathered} \dot{E} = -A^TE - EA - EBB^TE \\ E(t_f) = -Q. \end{gathered} \tag{16}$$

The optimal control u^* is consequently a feedback control given by

$$u^*(t) = B^TE^T(t)v(t) \tag{17}$$

and integrating backwards the equation 16 one obtains the initial condition

$$\eta^{*T}(0) = v_0^{*T}E(0). \tag{18}$$

4. **Energy-minimal transfers computations around the point L_1.** Let us now consider the non-linear control system in the vicinity of L_1

$$\begin{aligned}\ddot{x} - 2\dot{y} - (1+2c_2)x &= \frac{\partial}{\partial x}\sum_{n\geq 3} c_n\rho^n P_n(\frac{x}{\rho}) + u_1 \\ \ddot{y} + 2\dot{x} + (c_2-1)y &= \frac{\partial}{\partial y}\sum_{n\geq 3} c_n\rho^n P_n(\frac{x}{\rho}) + u_2 \\ \ddot{z} + c_2 z &= \frac{\partial}{\partial z}\sum_{n\geq 3} c_n\rho^n P_n(\frac{x}{\rho}) + u_3\end{aligned} \tag{19}$$

which is derived from 3. As we mentionned previously, the invariant manifolds of the linear dynamics near L_1 are good approximations of the invariant manifolds that exist in the non-linear dynamics case. This is why we keep the definition of $W^s_+(S^3_h)$ and $W^u_+(S^3_h)$ we gave in section 2. For the sake of simplicity, we investigate the problem of computing energy-minimal transfers from the submanifold $W^s_\alpha(S^3_h)$ where α is a stricly positive constant and p_1 is set to α, to $W^u_\alpha(S^3_h)$ where q_1 is set to α. Once again, the transfer time t_f is fixed and no constraints on the control bound are considered. Keeping the notation $v = (q,p) = (x,y,z,p_x,p_y,p_z)$, the optimal control problem we are investigating writes

$$\begin{array}{c}\dot{v} = F_0(v) + \sum_{i=1}^3 u_i F_i(v) \\ \min_{u(.)\in\mathbb{R}^2}\int_0^{t_f} u_1^2 + u_2^2 + u_3^2 dt \\ v(0)\in W^s_\alpha(S^3_h),\;\; v(t_f)\in W^u_\alpha(S^3_h)\end{array} \tag{20}$$

where

$$F_0(v) = \begin{pmatrix} p_1 + q_2 \\ p_2 - q_1 \\ p_3 \\ \frac{\partial}{\partial q_1}\sum_{n\geq 2} c_n\rho^n P_n(\frac{q_1}{\rho}) + p2 - q_1 \\ \frac{\partial}{\partial q_2}\sum_{n\geq 2} c_n\rho^n P_n(\frac{q_1}{\rho}) - p1 - q_2 \\ \frac{\partial}{\partial q_3}\sum_{n\geq 2} c_n\rho^n P_n(\frac{q_1}{\rho}) \end{pmatrix},\;\; F_i(v) = \frac{\partial}{\partial p_i},\;\; \text{i = 1, 2, 3.}$$

Control theory provides powerfull tools to study optimal solutions from the geometric point of view. First, from the Pontryagin's Maximum Principle, see [10],optimal solutions are found among extremal curves $(v,\eta)\in T^*\mathbb{R}^6$ solutions of the system

$$\dot{v} = \frac{\partial H}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial H}{\partial v} \tag{21}$$

where H is the pseudo-Hamiltonian function

$$\begin{aligned}H(v,\eta,u) &= <\eta, F_0(v)> + \sum_{i=1}^3 u_i <\eta, F_i(v)> + \eta^0(\sum_{i=1}^3 u_i^2) \\ &= H_0(v,\eta) + \sum_{i=1}^3 u_i H_i(v,\eta) + \eta^0(\sum_{i=1}^3 u_i^2)\end{aligned} \tag{22}$$

and η^0 is a non-positive constant. Moreover, the optimal control u^* satisfies the maximization condition

$$H(v,\eta,u^*) = \max_{w\in\mathbb{R}^3} H(v,\eta,v) \tag{23}$$

that involves $H_i = -2p^0 u_i$. In the normal case $p^0 \neq 0$, one can normalize p^0 to -1/2 which gives $H_i = u_i$ so that, substituting in 22, H becomes the true Hamiltonian

$$H_r(v,\eta) = H_0(v,\eta) + \frac{1}{2}(\eta_4^2 + \eta_5^2 + \eta_6^2) \tag{24}$$

and equation 21 can be written

$$(\dot{v},\dot{\eta}) = \overrightarrow{H_r}(v,\eta) \tag{25}$$

where $\overrightarrow{H_r}$ is the Hamiltonian vectorfield associated with H_r. Finally, the following final transversality condition

$$\eta(t_f) \perp T_{v(t_f)} W^u_\alpha(S^3_h). \tag{26}$$

has to be fullfiled. Note that the final conditions $v(t_f) \in W^u_\alpha(S^3_h)$ and $\eta(t_f) \perp T_{v(t_f)} W^u_\alpha(S^3_h)$ can be written $\gamma(v(t_f)) = 0$ where the function γ is defined by

$$\gamma : x \in \mathbb{R}^{12} \to \begin{pmatrix} x_1 - \alpha \\ x_4 \\ \frac{\omega_1}{2}(x_2^2 + x_5^2) + \frac{\omega_2}{2}(x_3^2 + x_6^2) - h \\ < (x_7, \dots, x_{12}), h_1 > \\ < (x_7, \dots, x_{12}), h_2 > \\ < (x_7, \dots, x_{12}), h_3 > \end{pmatrix} \tag{27}$$

and the family $\{h_1,\ h_2,\ h_3\}$ spans the 3-dimensional tangent space $T_{v(t_f)} W^u_\alpha(S^3_h)$. The Maximum principle is a necessary optimality condition and to get a necessary sufficient one we have to introduce the notion of conjugate time, see [5].

Definition 4.1. Consider a smooth manifold M of dimension n and an Hamiltonian system $\dot{X} = \overrightarrow{H}(X(t))$ where $X = (q,p) \in T^*M$ is written in local coordinates. The variational equation

$$\dot{\delta} X(t) = d\overrightarrow{H}(X).\delta X(t) \tag{28}$$

is called the ***Jacobi equation*** along X. One calls a ***Jacobi field*** a nontrivial solution $J(t) = (\delta q(t), \delta p(t))$ of the Jacobi equation along X and it is said to be ***vertical*** at time t if $\delta q(t) = 0$. A time t_c is said to be ***geometrically conjugate*** if there exists a Jacobi field vertical at 0 and t_c. In which case, $q(t_c)$, is said to be ***conjugate*** to $q(0)$.

In order to give a geometric characterization of conjugate times, let us define the so-called exponential mapping.

Definition 4.2. Let be $q_0 \in M$ and $t \in [0, t_f]$. One defines the ***exponential mapping*** by

$$\exp_{q_0,t} : p_0 \longrightarrow q(t, q_0, p_0)$$

where $q(t, q_0, p_0)$ is the projection on the phase space of the unique trajectory X of $\overrightarrow{H}$ satisfying $X(0, q_0, p_0) = (q_0, p_0)$

Let $\exp_t(\overrightarrow{H})$ be the flow of $\overrightarrow{H}$. The following proposition results from a geometrical interpretation of the Jacobi equation [6].

Theorem 4.3. *Let be $q_0 \in M$, $L_0 = T^*_{q_0}M$ and $L_t = \exp_t(\overrightarrow{H})(L_0)$. Then L_t is a Lagrangian submanifold of T^*M whose tangent space is spanned by Jacobi fields starting from L_0. Moreover $q(t_c)$ is geometrically conjugate to q_0 if and only if $\exp_{q_0,t_c}$ is not an immersion at p_0.*

Under generic assumptions, the following theorem connects the notion of conjugate time and the local optimality of extremals, see [7, 1, 12].

Theorem 4.4. *Let t_c^1 be the first conjugate time along z. The trajectory $q(.)$ is locally optimal on $[0, t_c^1)$ in L^∞ topology; if $t > t_c^1$ then $q(.)$ is not locally optimal on $[0, t]$.*

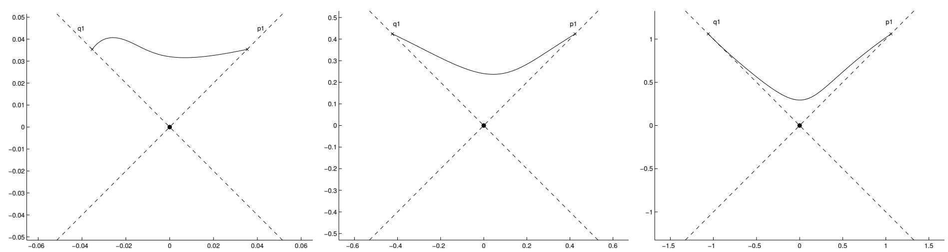
When the final target is a regular submanifold M_1, the notion of conjugate time is generalized as follows and the theorem 4.4 still holds.

Definition 4.5. Denote $M_1^\perp = \{(q, p),\ q \in M_1, p \perp T_q M_1\}$. Then a time t_{foc} is said to be a ***focal*** *time* if there exists a Jacobi field $J = (\delta q, \delta p)$ such that $\delta q(t_{\text{foc}}) = 0$ et $J(t_{\text{foc}})$ is tangent to $M_1^\perp$.

Therefore, evaluating the local optimality of an extremal curve consists in comparing the transfer time and the first conjugate time along the extremal. The numeric methods we use to compute energy-minimal trajectories from $W_\alpha^s(S_h^3)$ to $W_\alpha^u(S_h^3)$ are implemented in the COTCOT, see [6]. Fixing an initial condition z_0 and integrating numerically the Hamiltonian vectorfield $\overrightarrow{H_r}$, we can, using a Newton-type algorithm, find a zero of the shooting function $S:\ \eta_0 \to \gamma(v(t_f, v_0, \eta_0))$ and henceforce compute an extremal curve solution of the Maximum principle. We initialize the Newton algorithm using the initial adjoint vector η_0^* corresponding to the optimal trajectory of the linearized case which is numerically computed following the method described in the section 3. Since the target $W_\alpha^u(S_h^3)$ is a 3-dimensional submanifold of $\mathbb{R}^6$, the first focal point is evaluated integrating backward the derivative $\overrightarrow{H_r}$. The submanifold $W_\alpha^u(S_h^3)^\perp$ being 6-dimensional in $\mathbb{R}^{12}$, so is the tangent space $T_{v(t_f),\eta(t_f)} W_\alpha^u(S_h^3)^\perp$. We consequently consider the 6-dimensional vector space spanned by the Jacobi fields $J_i(t) = (\delta v_i, \delta \eta_i)$ for i=1,…,6 such that $T_{v(t_f),\eta(t_f)} W_\alpha^u(S_h^3)^\perp = \text{Span}\{J_i(0),\ 1 \le i \le 6\}$. A time t is then a focal time if $\text{rank}(\delta v_1(-t), \dots, \delta v_6(-t))$ is lower than or equal to 5.

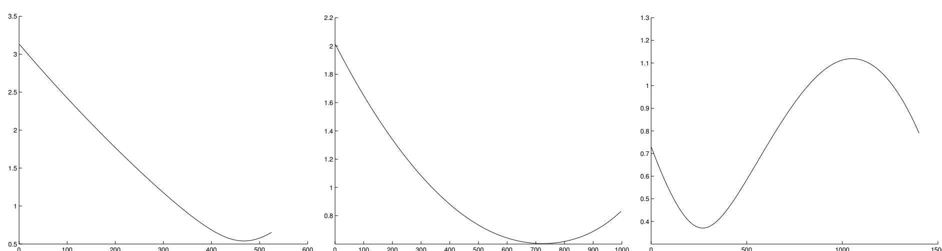
In the numerical computations we perform, the spacecraft's mass is assumed to be 350 kg, μ is set to 0.012153, h is set to 1.58 and we make the parameter α vary in order to evaluate how far the invariant manifolds structure extends. In figures 2 and 3 are respectively displayed the projections on the (p_1, q_1)-plane of the energy minimal extremal trajectories (where the axes are tilted by 45 degrees to be coherent with figures from [9]) and the norm of the corresponding extremal control for different values of α. In figure 4, the transfer time and the first focal time corresponding to the same values of α are compared.

Our numerical results show the efficiency of initializing the simple shooting method using the initial adjoint vector η_0^* corresponding to the linear case. We thus compute extremal trajectories, whose local optimality is ensured by the second order condition, from $W_\alpha^s(S_h^3)$ to $W_\alpha^u(S_h^3)$ for values of α higher than 1.5. Note that this treeshold, sending back in the initial resticted 3-body coordinates, approximatively corresponds to 80 percent of the distance from the collinear point L_1 to the Moon. Expressing the norm of extremal control in units of force, we can deduce from figure 3 that the maximal thrust needed to reach the unstable manifold from the stable one is contained between 0.4 and 0.1 Newton. It appears in figure 2 that, as we expected, the free dynamics in the invariant manifolds plays an important role in energy-minimal transfers in the Earth-Moon system. Indeed, the bigger is α, the longer the spacecraft follows the stable manifold before reaching the instable one; however bigger seems to be the maximal thrust because of the closeness with L_1. Let us point out that, for a fixed value of α, we can, in a certain extent, reduce the maximal thrust needed to achieve the transfer by making the transfer time t_f increase. In figure 5, we display the evolution of the norm of the extremal control for different values of t_f and $\alpha = 1$. However, computations show that when the transfer time is too long, the condition $t_f < t^1_{\text{foc}}$ is no more satisfied whereby extremal trajectories lose their optimality.



(a) $\alpha = 0.05$ (b) $\alpha = 0.6$ (c) $\alpha = 1.5$

FIGURE 2. Projections of energy-minimal trajectories on the (p_1, q_1)-plane (axes tilted 45 degrees).



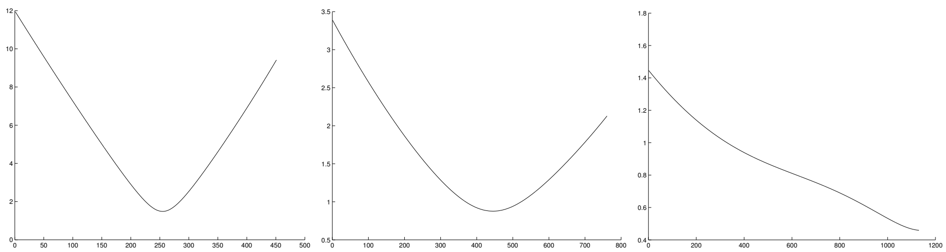
(a) $\alpha = 0.05$ (b) $\alpha = 0.6$ (c) $\alpha = 1.5$

FIGURE 3. Norm of extremal controls.

5. **Conclusion.** Basing on previous studies concerning dynamics in the restricted 3-body problem, we provide in this article an efficient procedure for initializing a numeric indirect method in optimal control and computing energy-minimal transfers in the vicinity of the Lagrangian point L_1. We show that, from the optimal control theory point of view, invariant manifolds in the equilibrium region play an important role for envisioning transfers from the Earth's gravity area to the Moon's gravity one. Besides, this work is a good example of a powerfull application of geometric control theory combined with numerics methods.

α	0.05	0.6	1.5
t_f	0.524	0.996	1.4
t^1_{foc}	no focal time in $[0,2t_f]$	1.6	no focal time in $[0,2t_f]$

FIGURE 4. Comparision between the transfer time t_f and the first focal time t^1_{foc} along extremals.



(a) t_f=0.45 (b) t_f=0.76 (c) t_f=1.13

FIGURE 5. Norm of extremal control corresponding to different transfer time for α=1.

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