Hw 2, Math 305, Spring 2018

Last name, first name, ID number : Score:

1. Let X and Y be two random variables and $\rho(X,Y)$ be the dimensionless correlation coefficient between X and Y. Show that $-1 \leq \rho(X,Y) \leq 1$.
   Solution: Let's define $\mu_X = \mathrm{E}(X)$, $\mu_Y = \mathrm{E}(Y)$, $\sigma_X^2 = \mathrm{Var}(X)$, $\sigma_Y^2 = \mathrm{Var}(Y)$, $U = \frac{X-\mu_X}{\sigma_X} \sim \mathcal{N}(0,1)$, $V = \frac{Y-\mu_Y}{\sigma_Y} \sim \mathcal{N}(0,1)$ and $W = (U - \frac{\mathrm{E}(UV)}{E(V^2)}V)^2$. Since W is a positive random variable, its expected value is positive. Calculating, we find $\mathrm{E}(W) = \mathrm{E}(U^2) - \frac{\mathrm{E}(UV)^2}{\mathrm{E}(V^2)} \geq 0$ so $\mathrm{E}(U^2)\mathrm{E}(V^2) \geq \mathrm{E}(UV)^2$. Notice that $\mathrm{Var}(U) = \mathrm{E}(U^2) - \mathrm{E}(U)^2 = \mathrm{E}(U^2) = 1$. Similarly $\mathrm{E}(V^2) = 1$. Thus $1 \geq \mathrm{E}(UV)^2$ so $-1 \leq \mathrm{E}(UV) \leq 1$. Moreover $\mathrm{E}(UV) = \mathrm{Cov}(U,V) = \frac{1}{\sigma_X\sigma_Y}\mathrm{Cov}(X,Y) = \rho(X,Y)$. Therefore $-1 \leq \rho(X,Y) \leq 1$.

2. Let X and Y be two random variables.

   (a) Show that $E(XY)^2 \leq E(X^2)E(Y^2)$.
   Solution: Set $U = (X - \frac{\mathrm{E}(XY)}{\mathrm{E}(Y^2)}Y)^2$. Since X is a positive random variable, its expected value is positive. Calculating, we find $\mathrm{E}(U) = \mathrm{E}(X^2) - \frac{\mathrm{E}(XY)^2}{\mathrm{E}(Y^2)} \geq 0$. Therefore $E(XY)^2 \leq E(X^2)E(Y^2)$.

   (b) Show that $\mathrm{cov}(X,Y)^2 \leq \mathrm{Var}(X)\mathrm{Var}(Y)$.
   Solution: Let's define $\mu_X = \mathrm{E}(X)$, $\mu_Y = \mathrm{E}(Y)$, $U = X - \mu_X$, $V = Y - \mu_Y$ and $W = (U - \frac{\mathrm{E}(UV)}{\mathrm{E}(V^2)}V)^2$. Since W is a positive random variable, its expected value is positive. Calculating, we find $\mathrm{E}(W) = \mathrm{E}(U^2) - \frac{\mathrm{E}(UV)^2}{\mathrm{E}(V^2)} \geq 0$ so $\mathrm{E}(U^2)\mathrm{E}(V^2) \geq \mathrm{E}(UV)^2$. Notice that $\mathrm{Var}(X)\mathrm{Var}(Y) = E(U^2)E(V^2)$ and $\mathrm{cov}(X,Y)^2 = \mathrm{E}(UV)^2$. Therefore $\mathrm{cov}(X,Y)^2 \leq \mathrm{Var}(X)\mathrm{Var}(Y)$.

3. One deterministic model of population growth with limitedresources is the logistic model: if x_t is the population at year $t \in \mathbb{N}$, r_0 is the population per capita growth rate when resouces are unlimited and k is the carrying capacity, that-is-to-say the largest population size that the environment can maintain, yjen the model is

$$x_{t+1} = x_t\Big(1 + r_0(1 - \frac{x_t}{k})\Big).$$

   Suppose we want to change it to a stochastic model for X_t, the random variable giving the population at time t.

   - How might we introduce random yearly variation in the per capita growth rate? Does it represent environmental or demographic stochasticity?
     Solution: We can replace r_0 by $R_0 \sim (r_0, \sigma^2)$ where we should decide/make an assumption about what the variance σ^2 is. It represents demographic stochasticity.
   - Suppose droughts every 5 years (on average) reduce capacity in half. How would we introduce this into the model? Does it represent environmental or demographic stochasticity?
     Solution: We can replace k by a discrete random variable K whose p.f. is

$$P(K = k) = 0.8,\ P(K = k/2) = 0.2.$$

     It represents environmental stochasticity.

4. Find the solution of the difference equation $u_{n+2} - 3u_{n+1} + 2u_n = 0$ subject to boundary conditions $u_0 = 0$ and $u_5 = 62$.
   Solution: let's look for a solution in the form $u_n = \lambda^n$. Plugging in the equation, we get

$$\lambda^{n+2} - 3\lambda^{n+1} + 2\lambda^n = 0$$

   that is

$$\lambda^n(\lambda^2 - 3\lambda + 2) = 0.$$

   We have either $\lambda = 0$ which does not satisfy the boundary conditions, or $\lambda^2 - 3\lambda + 2 = 0$ which leads to $\lambda = 1$ or $\lambda = 2$. The general solution of the difference equation is $u_n = a + b2^n$. We want $u_0 = a + b = 0$ so $a = -b$ and $u_n = b(2^n - 1)$. Also, we want $u_5 = b(21 - 1) = 62$ so $b = 2$. Therefore, the solution is $u_n = -2 + 2^{n+1}$.

5. Suppose a random walk is defined by

$$X_n = X_{n-1} + Z_n,\ n = 1,\ 2,\ 3, \ldots$$

where the random variables Z_n are independent and identically distributed with $P(Z_n = 1) = p$, $P(Z_n = 0) = r$, $P(Z_n = -1) = q$ and $p + r + p = 1$. Assume the values of the random variables X_n are $0,\ 1, \ldots, N$. Let u_n the probability of reaching N before 0 when $X_0 = n$.

(a) Set up a difference equation for u_n with the appropriate boundary conditions at 0 and N.
Solution: Define the event $A_n,\ R,\ L$ and S by

$$\begin{aligned} A_n &= \text{reaching } N \text{ before 0 when } X_0 = n, \\ R &= \text{move right on 1st move}, \\ L &= \text{move left on 1st move}, \\ S &= \text{stay put on 1st move}. \end{aligned}$$

Thus we have

$$\begin{aligned} u_n &= P(A_n) \\ &= P(A_n|R)P(R) + P(A_n|L)P(L) + P(A_n|S)P(S) \\ &= u_{n+1}p + u_{n-1}q + u_n r. \end{aligned}$$

Hence

$$(1-r)u_n = u_{n+1}p + u_{n-1}q$$

that is

$$u_n = u_{n+1}p' + u_{n-1}q'$$

where $p' = \frac{p}{1-r}$ and $q' = \frac{q}{1-r}$. Note that $p' + q' = 1$.

(b) Show that the expressions for the probabilities are the same as those for the simple random walk described in section 7.3.
Solution: The difference equation for u_n is the same as the recurrence equation (7.8) p 320 in the textbook, with the same boundary conditions $u_0 = 0$ and $u_N = 1$. Therefore, the solution is the same: if $p \neq q$ (that is $p' \neq q'$) then

$$u_n = \frac{1-(p'/q')^n}{1-(p'/q')^N} = \frac{1-(p/q)^n}{1-(p/q)^N}.$$

6. Find the general solution of the differential equation $y'(t) = 2y(t) + e^{4t}$.
Solution: Using the method of the integrating factor, we find $y(t) = e^{2t}\int e^{-2t}e^{4t}dt = e^{2t}\left(\frac{e^{2t}}{2} + C\right) = \frac{e^{4t}}{2} + Ce^{2t}$.
Sample exam 2 Wilson

7. Assume $N(t)$ is a population at time t. Assume there are no births and

$$P(1 \text{ death in } (t, t+h)|N(t) = n) = n\delta h + o(h)$$

$$P(\text{more than 1 death in } (t, t+h)|N(t) = n) = o(h)$$

(a) Determine $P(0 \text{ death in } (t, t+h)|N(t) = n)$.
Solution: $P(0 \text{ death in } (t, t+h)|N(t) = n) = 1 - n\delta h + o(h)$.

(b) Find the Kolmogorov differential equation for $p_n(t) = P(N(t) = n)$.
<u>Solution</u>: Note that $p_n(t) = P(N(t) = n)$ for $n \in \{n_0, n_0 - 1, \ldots, 0\}$. We have

$$p_n(t+h) = p_{n+1}(t)\big((n+1)\delta h + o(h)\big) + p_n(t)\big(1 - n\delta h + o(h)\big) + o(h)$$

hence

$$\frac{p_n(t+h) - p_n(t)}{h} = p_{n+1}(t)(n+1)\delta - p_n(t)n\delta + \frac{o(h)}{h}.$$

hence

$$p_n'(t) = (n+1)\delta p_{n+1}(t) - n\delta p_n(t).$$

(c) Without proof, what should the p.d.f. of the interevent time be, when assuming that $N(0) = n_0$?
<u>Solution</u>: It should be the p.d.f. of an exponential distribution with rate constantly decreasing as the population decreases, that is $X_1 \sim n_0\delta e^{-n_0\delta t}$, $X_2 \sim (n_0 - 1)\delta e^{-(n_0-1)\delta t}$, ect...