Hw 2, Math 472, Spring 2018

Last name, first name, ID number : Score:

1. A bottling machine can be regulated so that it discharges an average of μ ounces per bottle. It has been observed that the amount of fill dispensed by the machine is normally distributed with $\sigma = 1$ ounce. A sample of $n = 9$ filled bottles is randomly selected from the output of the machine on a given day (all bottled with the same machine setting) and the ounces of fill are measured for each.

   (a) Find the probability that the sample mean will be within .3 ounces of the true mean for the chosen machine setting? (Use a probability table).
   Solution: Let $Y_1, \dots, Y_9$ denote the ounces of fill to be observed. We know that the random variables $Y_i,\ 1 \le i \le 9$ are all normally distributed with mean μ and variance $\sigma^2 = 1$. According to corollary 5.6.2, the sample mean $\bar{Y}$ is normally distributed with mean μ and variance 1/9. We want to find

$$\begin{aligned}P(|\bar{Y} - \mu| \le 0.3) &= P(-0.3 \le \bar{Y} - \mu \le 0.3)\\ &= P\Big(\frac{-0.3}{1/\sqrt{9}} \le \frac{\bar{Y} - \mu}{1/\sqrt{9}} \le \frac{0.3}{1/\sqrt{9}}\Big)\\ &= P\big(-0.9 \le Z \le 0.9\big) \text{ where } Z \sim \mathcal{N}(0,1)\\ &\approx 0.6318 (\text{from a probability table.})\end{aligned}$$

   (b) How many observations should be included in the sample if we wish $\bar{Y}$ to be within 0.3 ounces of μ with probability 0.95? (Use a probability table.)
   Solution: Now we want

$$P(|\bar{Y} - \mu| \le 0.3) = P(-3 \le \bar{Y} - \mu \le 0.3) = 0.95.$$

   Dividing each term of the inequality by $\sigma_{\bar{Y}} = 1\sqrt{n}$, we have

$$P(-0.3\sqrt{n} \le Z \le 0.3\sqrt{n}) = 0.95, \text{ where } Z \sim \mathcal{N}(0,1).$$

   By using a probability table, we find

$$P(-1.96 \le Z \le 1.96) = 0.95.$$

   It must follow that $0.3\sqrt{n} = 1.96$ or, equivalently, $n = \left(\frac{1.96}{0.3}\right)^2 \approx 42.68$. Practically, we should include 42 observations.

2. Suppose that a particle is located at the origin of the 3D-space at $t = 0$. Denote (X, Y, Z) the coordinates of the particle at any time $t > 0$. The random variables X, Y and Z are assumed to be i.i.d. and, at any time $t > 0$, each of them has the normal distribution with mean 0 and variance $\sigma^2 t$. Find the probability that, at $t = 2$, the particle will lie within a sphere whose center is at the origin and whose radius is 4σ.
   Solution:At $t = 2$, the variables X, Y and Z are independent from the normal distribution $\mathcal{N}(0, 2\sigma^2)$ hence the variables $\frac{X}{\sqrt{2}\sigma}$, $\frac{Y}{\sqrt{2}\sigma}$ and $\frac{Z}{\sqrt{2}\sigma}$ are independent from the standard normal distribution and the variable $\frac{X^2+Y^2+Z^2}{2\sigma^2}$ follows a χ^2-distribution with 3 degrees of freedom. Thus

$$\begin{aligned}P(\text{ At } t = 2, \text{ particle lies within a sphere centered at origin with radius is } 4\sigma) &= P(X^2 + Y^2 + Z^2 \le 16\sigma^2)\\ &= P(\frac{X^2 + Y^2 + Z^2}{2\sigma^2} \le 8)\\ &= \Phi_{\chi^2(3)}(8)\\ &\approx 0.954.\end{aligned}$$

3. Suppose that $X_1, \dots, X_n$ form a random sample from the normal distribution with mean μ and variance σ^2. Assume that the sample size is $n = 16$. Determine the values of

(a) $P\Big(\frac{\sigma^2}{2} \le \frac{1}{n}\sum_{i=1}^n (X_i-\mu)^2 \le 2\sigma^2\Big)$
Solution: Notice that, for all $1 \le i \le 16$, $\frac{(X_i-\mu)^2}{\sigma^2} \sim \mathcal{N}(0,1)$ so $\sum_{i=1}^n \frac{(X_i-\mu)^2}{\sigma^2} \sim \chi^2(16)$.Hence, when $n = 16$,

$$\begin{aligned} P\Big(\frac{\sigma^2}{2} \le \frac{1}{n}\sum_{i=1}^n (X_i-\mu)^2 \le 2\sigma^2\Big) &= P\Big(\frac{n}{2} \le \sum_{i=1}^n \frac{(X_i-\mu)^2}{\sigma^2} \le 2n\Big) \\ &= \Phi_{\chi^2(16)}(32) - \Phi_{\chi^2(16)}(8) \\ &\approx 0.939 \end{aligned}$$

(b) $P\Big(\frac{\sigma^2}{2} \le \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X}_n)^2 \le 2\sigma^2\Big)$.
Solution: By theorem 8.3.1, $\sum_{i=1}^n \frac{(X_i-\bar{X}_n)^2}{\sigma^2} \sim \chi^2(n-1)$.Hence, when $n = 16$,

$$\begin{aligned} P\Big(\frac{\sigma^2}{2} \le \frac{1}{n}\sum_{i=1}^n (X_i-\bar{X}_n)^2 \le 2\sigma^2\Big) &= P\Big(\frac{n}{2} \le \sum_{i=1}^n \frac{(X_i-\bar{X}_n)^2}{\sigma^2} \le 2n\Big) \\ &= \Phi_{\chi^2(15)}(32) - \Phi_{\chi^2(15)}(8) \\ &\approx 0.918 \end{aligned}$$

4. The tensile strength for a type of wire is normally distributed with unknown mean μ and unknown variance σ^2. Six pieces of wire are randomly selected from a large roll; Y_i, the tensile strength for portion i, is measured for $1 \le i \le 6$. The population mean μ and variance σ^2 can be estimated by $\bar{Y}$ and S^2, respectively. Because $\sigma^2_{\bar{Y}} = \frac{\sigma^2}{n}$ (corollary 5.6.2), it follows that $\sigma^2_{\bar{Y}}$ can be estimated by $\frac{S^2}{n}$. Find the approximate probability that $\bar{Y}$ will be within $\frac{2S}{\sqrt{n}}$ of the true population mean μ.
Solution: We want

$$P\Big(-\frac{2S}{\sqrt{n}} \le \bar{Y}-\mu \le \frac{2S}{\sqrt{n}}\Big) = P\Big(-2 \le \sqrt{n}\frac{\bar{Y}-\mu}{S} \le 2\Big) = P(-2 \le T \le 2)$$

where T has a t-distribution with, in this case, $n-1$ df. From a probability table, we can see that

$$P(-2.015 \le T \le 2.015) = 0.90.$$

Therefore, the probability that $\bar{Y}$ is within 2 estimated standard deviation of μ is slightly less than 0.9.

5. Suppose that Y_1, Y_2 and Y_3 form a random sample from an exponential distribution with parameter θ. Consider the estimators $\hat{\theta}_1 = Y_1$, $\hat{\theta}_2 = \frac{1}{2}(Y_1+Y_2)$ and $\hat{\theta}_3 = \bar{Y}$.

(a) Show that $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ are unbiased estimators for $\frac{1}{\theta}$.
Solution: $E(\hat{\theta}_1) = E(Y_1) = 1/\theta$, $E(\hat{\theta}_2) = \frac{1}{2}(E(Y_1)+E(Y_2)) = 2/2\theta = 1/\theta$ and $E(\hat{\theta}_3) = \frac{1}{3}(E(Y_1)+E(Y_2)+E(Y_3)) = 3/3\theta = 1/\theta$ so $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ are unbiased estimators for $\frac{1}{\theta}$.

(b) Which of $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$ has the least variance?
Solution: $\mathrm{Var}(\hat{\theta}_1) = \mathrm{Var}(Y_1) = 1/\theta^2$, $\mathrm{Var}(\hat{\theta}_2) = (1/4).(\mathrm{Var}(Y_1)+\mathrm{Var}(Y_2)) = 1/2\theta^2$ and $\mathrm{Var}(\hat{\theta}_3) = (1/3).(\mathrm{Var}(Y_1)+\mathrm{Var}(Y_2))+\mathrm{Var}(Y_3)) = 1/3\theta^2$ so $\hat{\theta}_3$ has the smallest variance, that is the smallest MSE and, therefore, is the best estimator.

6. Suppose that Y_1, Y_2 and Y_3 form a random sample from a uniform distribution on the interval $(\theta, \theta+1)$.

(a) Calculate the bias for $\bar{Y}$ as an estimator of θ.
Solution: $\mathrm{Bias}(\bar{Y}) = E(\bar{Y}) - \theta = (2\theta+1)/2 - \theta = 1/2$.

(b) Calculate the bias for $\bar{Y} - \frac{1}{2}$ as an estimator of θ.
Solution: $\mathrm{Bias}(\bar{Y} - 1/2) = E(\bar{Y}) + 1/2 - \theta = 0$.

(c) Calculate the MSE for $\bar{Y}$ as an estimator of θ.
Solution: $\mathrm{MSE}(\bar{Y}) = \mathrm{Var}(\bar{Y}) + (\mathrm{Bias}(\bar{Y}))^2 = \frac{1}{9}\frac{3}{12} + \frac{1}{4} = \frac{5}{18}$.