Hw 3, Math 305, Spring 2018

Last name, first name, ID number : Score:

1. If $f(t)$ is a continuous function, how can we define a random process $X(t)$ continuous in t which is normal for each t with $E(X(t)) = f(t)$ and $\text{Var}(X(t)) = 4t$?
   Solution: Set $X(t) = f(t) + 2W(t)$. Thus $X(t) \sim \mathcal{N}(f(t), f(t))$

2. Is the matrix $\begin{pmatrix} 0.2 & 0.2 & 0.6 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.6 & -0.1 \end{pmatrix}$ a probability transition matrix?
   Solution: No, not all entries are positive.

3. Assume that the transition matrix for a Markov chain with states $\{s_1,\ s_2,\ s_3,\ s_4\}$ is $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.2 & 0.6 & 0.2 & 0 \end{pmatrix}$.

   (a) What are the absorbing states?
   Solution: The absorbing states are s_1 and s_2 since the diagonal entries in the first and second rows of the transition matrix equal 1.

   (b) What is the probability that s_4 is absorbed by s_1?
   Solution: We have $Q = \begin{pmatrix} 0 & 0.5 \\ 0.2 & 0 \end{pmatrix}$ so $I - Q = \begin{pmatrix} 1 & -0.5 \\ -0.2 & 0 \end{pmatrix}$, $\Phi = (I - Q)^{-1} = \frac{10}{9}\begin{pmatrix} 1 & 0.5 \\ 0.2 & 1 \end{pmatrix}$ and $\Pi = \Phi R = \frac{1}{9}\begin{pmatrix} 1 & 8 \\ 2 & 7 \end{pmatrix}$. Therefore, the probability that s_4 is absorbed by s_1 is $\frac{2}{9}$.

   (c) What is the expected time for s_3 to be absorbed?
   Solution: We have $\Phi\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{4}{3} \end{pmatrix}$. Thus the expected time for s_3 to be absorbed is $\frac{5}{3}$.

4. Assume that the transition matrix for a Markov chain with states $\{s_1,\ s_2,\ s_3\}$ is $P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$.

   (a) Show that the vector $v = [0.4\ 0.4\ 0.2]$ is a left eigenvector for P with the eigenvalue 1.
   Solution: We have $vP = v$ so v is a left eigenvector for P with the eigenvalue 1.

   (b) Find the stationary distribution.
   Solution: Since v is a left eigenvector for P with the eigenvalue 1 and the sum of the entrie of v is 1, then v is the stationary distribution.

   (c) What does $[0.4\ 0.4\ 0.2] \cdot P^n$ represent? What does $[0.4\ 0.4\ 0.2] \cdot P^n$ converge to as n approaches ∞?
   Solution: The vector $[0.4\ 0.4\ 0.2]\cdot P^n$ represent the distribution of X_n if the initial distribution is $[0.4\ 0.4\ 0.2]$. Since $[0.4\ 0.4\ 0.2]$ is the stationary distribution, the vector $[0.4\ 0.4\ 0.2] \cdot P^n$ is constant equal to $[0.4\ 0.4\ 0.2]$ for all n so $[0.4\ 0.4\ 0.2] \cdot P^n$ converge to $[0.4\ 0.4\ 0.2]$ as n approaches ∞.

5. Consider the logistic equation
$$\frac{dx}{dt} = rx(1 - \frac{x}{K}), \quad x(0) = x_0.$$
   Add stochasticity to the model by assuming that the inverse of the carrying capacity contains noise of the form $\sigma W(t)$ (in other words, replace $\frac{1}{K}$ by $\frac{1}{K} + \sigma W(t)$). Write the random model in the form of an Itô Process.
   Solution: Replacing $\frac{1}{K}$ by $\frac{1}{K} + \sigma W(t)$), we get the Itô Process
$$dX(t) = rX(t)\Big(1 - \frac{X(t)}{K}\Big)dt - rX(t)^2\sigma dW(t).$$

6. In each year of a three-year degree course, a university student has probability p of not returning the following year, probability q of having the repeat the year and probability r of passing (where we assume that $p+q+r=1$). The states are: dropped out (s_1), graduated (s_2), is a third-year student (s_3), is a second-year student (s_4) and is a first-year student (s_5).

   (a) Find the transition matrix P, and the matrices Q and R.

   Solution: We find $P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ p & r & q & 0 & 0 \\ p & 0 & r & q & 0 \\ p & 0 & 0 & r & q \end{pmatrix}$ so $Q = \begin{pmatrix} q & 0 & 0 \\ r & q & 0 \\ 0 & r & q \end{pmatrix}$ and $R = \begin{pmatrix} p & r \\ p & 0 \\ p & 0 \end{pmatrix}$.

   (b) Show that $\frac{1}{a^3}\begin{pmatrix} a^2 & 0 & 0 \\ -ab & a^2 & 0 \\ b^2 & -ab & a^2 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   Solution: Straightforward.

   (c) Using the result of the question (b), find the fundamental matrix $\Phi = (I-Q)^{-1}$.

   Solution: We find $I - Q = \begin{pmatrix} 1-q & 0 & 0 \\ -r & 1-q & 0 \\ 0 & -r & 1-q \end{pmatrix}$. Apply the result of the question (b) with $a = 1-q$ and $b = -r$, we find $\Phi = (I-Q)^{-1} = \begin{pmatrix} \frac{1}{1-q} & 0 & 0 \\ \frac{r}{(1-q)^2} & \frac{1}{1-q} & 0 \\ \frac{r^2}{(1-q)^3} & \frac{r}{(1-q)^2} & \frac{1}{1-q} \end{pmatrix}$

   (d) Find the student's chance of graduating if they are in years 1, 2 and 3. Solution: We want to calculate the probabilities of s_3, s_4 and s_5 to be absorbed by s_2. We find

$$\Pi = \Phi R = \frac{1}{(1-q)^3}\begin{pmatrix} p(1-q)^2 & r(1-q)^2 \\ pr(1-q) + p(1-q)^2 & r^2(1-q) \\ pr^2 + pr(1-q) + p(1-q)^2 & r^3 \end{pmatrix}$$

   so the student's chance of graduating if they are in years 1, 2 and 3 are $\frac{r^3}{(1-q)^3}$, $\frac{r^2}{(1-q)^2}$ and $\frac{r}{(1-q)}$, respectively.

   (e) Find the average number of years a first-year, second-year and third-year student will remain in university.

   Solution: Calculating $\Phi\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, we find that average number of years a first-year, second-year and third-year student will remain in university are $\frac{r^2+r(1-q)+(1-q)^2}{(1-q)^3}$, $\frac{r}{(1-q)^2} + 1 - q$ and $\frac{1}{1-q}$, respectively.