Hw 3, Math 472, Spring 2018

Last name, first name, ID number : Score:

1. Consider a single observation Y from an exponential distribution with mean θ where θ is unknown.

   (a) Find the probability density function of the random variable $U = \frac{Y}{\theta}$.
   Solution: Using the mgf or the cdf or a substitution, we find that $U \sim \mathcal{E}(1)$. Its pdf is the function

   $$f(u) = \begin{cases} e^{-u}, & u > 0 \\ 0, & u \leq 0. \end{cases}$$

   (b) Find a real number a such that $P(U < a) = 0.05$.
   Solution: the real number a is solution of $\int_0^a e^{-u} du = 0.05$. Integrating and solving for a, we find $a = -\ln(0.95) \approx 0.051$.

   (c) Find a real number b such that $P(U > b) = 0.05$.
   Solution: the real number b is solution of $\int_b^\infty e^{-u} du = 0.05$. Integrating and solving for b, we find $b = -\ln(0.05) \approx 2.99$.

   (d) Use the results of the previous questions to find an exact 90 % confidence interval for the parameter θ.
   Solution: From (b) and (c), we have

   $$P\Big(-\frac{Y}{\ln(0.05)} \leq \theta \leq -\frac{Y}{\ln(0.95)}\Big) = P\Big(-\ln(0.95) \leq U \leq -\ln(0.05)\Big) = 0.9.$$

   Thus $\Big(-\frac{Y}{\ln(0.05)}, -\frac{Y}{\ln(0.95)}\Big)$ is an exact 90 % confidence interval for the parameter θ.

2. Suppose that $X_1, \ldots, X_n$ form a random sample from the exponential distribution with parameter θ. Suppose that we want to test the hypotheses:

   $$H_0 : \theta \geq \theta_0,$$
   $$H_1 : \mu < \theta_0.$$

   Let $X = \sum_{i=1}^{n} X_i$ and let δ_c be the test that rejects H_0 if $X \geq c$.

   (a) Show that $\pi(\theta|\delta_c)$ is a decreasing function of θ.
   Solution: Since $X_1, \ldots, X_n$ form a random sample from the exponential distribution with parameter θ, the random variable $U = \theta X \sim \Gamma(n, 1)$. Thus, $\pi(\theta|\delta_c) = P(X \geq c) = P(U \geq \theta c) = 1 - G(\theta c)$ where G i the cdf of U. Since G is increasing and c must be positive (setting c equal to a negative value would led to always rejecting H_0), we conclude that $\pi(\theta|\delta_c)$ is a decreasing function of θ.

   (b) Find c in order to make δ_c have the size α_0.
   Solution: According to the result of the previous question, we have

   $$\alpha(\delta_c) = \sup_{\theta \geq \theta_0} \pi(\theta|\delta_c) = \sup_{\theta \geq \theta_0} 1 - G(\theta c) = 1 - G(\theta_0 c).$$

   Thus, for δ_c to have the size α_0, we need $1 - G(\theta_0 c) = \alpha_0$, that is $c = \frac{G^{-1}(1-\alpha_0)}{\theta_0}$.

   (c) Let $\theta_0 = 2$, $n = 1$ and $\alpha_0 = 0.1$. Find the precise form of the test δ_c and sketch its power function.
   Solution: Since the cdf of $\Gamma(1, 1)$ is $G(u) = 1 - e^{-u}$, we obtain the equation $e^{-2c} = 0.1$ whose solution is $c = -\frac{\ln(0.1)}{2} \approx 1.15$. The test procedure is : rejects H_0 if $X_1 \geq -\frac{\ln(0.1)}{2}$.

3. Suppose that $X_1, \ldots, X_{25}$ form a random sample from the normal distribution with mean μ and variance 1. Let μ_0 be a real number and assume that we want to test the hypotheses:

   $$H_0 : \mu = \mu_0$$
   $$H_1 : \mu \neq \mu_0$$

Consider a test procedure such that H_0 is to be rejected if $|\bar{X}_n - \mu_0| \geq c$. Determine the value of c such that the size of the test will be 0.05.

Solution:Denote δ the test procedure described in the statement. We want

$$\alpha(\delta) = P(|\bar{X}_{25} - \mu| \geq c | \mu = \mu_0) = P(|Z| \geq 5c) = 0.05.$$

where $Z \sim \mathcal{N}(0,1)$. Thus, we must have $c = \frac{\Phi^{-1}(0.975)}{5} \approx 0.392$.

4. Suppose that a random sample $X_1, \dots, X_8$ from the normal distribution with unknown mean μ and unknown variance σ^2 is observed and it is desired to test the following hypothese

$$H_0 : \mu = 0,$$
$$H_1 : \mu \neq 0.$$

Suppose that the sample data are such that $\sum_{i=1}^{8} X_i = -11.2$ and $\sum_{i=1}^{8} X_i^2 = 43.7$. If a symmetric $t-$test is performed at the level of significance 0.1 so that each tail of the critical region has probability 0.05, should the hypothesis H_0 be rejected or not?

Solution; Here, we must use the test procedure :

$$\text{Reject } H_0 \text{ if } |U| \geq T_7^{-1}(0.95)$$

where U is the test-statistic defined in Eq. (9.5.2) p. 576 of the text book. Calculating, we find $|U| \approx 1.98$ and $T_7^{-1}(0.95) \approx 1.895$ so H_0 should be rejected.

5. Suppose that $X_1, \dots, X_n$ form a random sample from the Poisson distribution with unknown mean λ. Let λ_0 and λ_1 be specified values such that $0 < \lambda_0 < \lambda_1$ and suppose that it is desired to test the following hypothese.

$$H_0 : \lambda = \lambda_0,$$
$$H_1 : \lambda = \lambda_1.$$

(a) Show that the value of $\alpha(\delta) + \beta(\delta)$ is minimized by a test procedure which rejects H_0 when $\bar{X}_n > c$.
Solution: By theorem 9.2.1 p 552, we know that $\alpha(\delta)+\beta(\delta)$ is minimized by a test procedure which rejects H_0 when the likelihood ratio exceeds 1. Calculating, we find that the likelihood ratio equals $e^{(\lambda_0-\lambda_1)n}\left(\frac{\lambda_1}{\lambda_0}\right)^{n\bar{X}_n}$. Since, $0 < \lambda_0 < \lambda_1$, it can be shown that the likelihood ratio exceeds 1 if and only if $\bar{X}_n \geq \frac{\lambda_1 - \lambda_0}{\ln(\lambda_1) - \ln(\lambda_0)}$.

(b) Find the value of c.
Solution: According to the previous question, $c = \frac{\lambda_1 - \lambda_0}{\ln(\lambda_1) - \ln(\lambda_0)}$.

(c) For $\lambda_0 = \frac{1}{4}$, $\lambda_1 = \frac{1}{2}$ and $n = 20$, determine the minimum value of $\alpha(\delta) + \beta(\delta)$ that can be attained.
Solution: we have $c = \frac{1/4}{\ln(1/2) - \ln(1/4)} \approx 0.361$. Thus,

$$\begin{aligned}\alpha(\delta) + \beta(\delta) &\approx P(\bar{X}_{20} > 0.361 | \lambda = \frac{1}{4}) + P(\bar{X}_{20} < 0.361 | \lambda = \frac{1}{2}) \\ &= P(Y_1 > 7.213 | Y_1 \sim \text{Poisson}(1/4)) + P(Y_2 < 7.213 | Y_2 \sim \text{Poisson}(1/2)) \\ &\approx 0.3536.\end{aligned}$$